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Goodness-of-fit Methods
Research Article

Finite-sample critical values for goodness-of-fit tests under the exponential distribution

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Abstract

Goodness-of-fit tests based on the likelihood ratio are widely used to assess whether a given probability distribution adequately describes observed data. However, certain likelihood ratio-based tests do not have known asymptotic distributions, making it necessary to rely on pre-tabulated critical values obtained through Monte Carlo simulations. A major limitation of this approach is that practitioners must generate additional critical values via simulations for sample sizes not explicitly tabulated, which restricts the applicability of these tests in practice. This study addresses this limitation by developing asymptotic critical value functions for likelihood ratio-based goodness-of-fit tests under the exponential distribution. The proposed methodology employs response surface analysis to express simulated critical values as functions of sample size, enabling rapid computation of finite-sample critical values without requiring extensive simulations. The response surface regressions are estimated using median regression, ensuring robustness to outliers and heteroskedasticity. Extensive Monte Carlo experiments demonstrate that the estimated asymptotic critical value functions provide highly accurate test sizes across a wide range of sample sizes.

Keywords: Asymptotic critical values · Goodness-of-fit tests · Median regression · Response surface methodology

1. Introduction

The exponential distribution is one of the most widely used continuous distributions in statistical analysis due to its simplicity, as it has only a single parameter. It is also recognized for its memoryless property, meaning that the probability of an event occurring in the next unit of time does not depend on how long it has been since the last event. This property simplifies calculations and predictions in reliability analysis, eliminating the need to track the age or history of a component or system. Consequently, the exponential distribution is a frequent choice in reliability studies, which focus on how long a component or system can function without failing. It is also employed in hydrology to analyze extreme values of variables such as monthly or annual maximum daily rainfall and river discharge volumes (Ritzema, 1994), and it finds further applications in queueing theory, quality control, medical research, and various other fields (Asmussen and Bladt, 1996; Willig, 1999; Santiago and Smith, 2013; Mukherjee et al., 2015; Tomitaka et al., 2016).

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When a specific distribution is assumed for a dataset without confirmation that it truly underpins the data-generating process, the resulting inferences can be misleading. This is because statistical methods are sensitive to the distributional assumptions of the data. Thus, erroneous assumptions can lead to invalid conclusions and different outcomes across studies. In reliability research, where precision is paramount, incorrect distributional assumptions can introduce harmful bias (Ossai et al., 2022). In recognition of these practical concerns, the literature features numerous tests for exponentiality, with early contributions by Massey Jr (1951), Anderson and Darling (1952, 1954) and others. The present study adds to this ongoing discussion by estimating asymptotic critical value functions for Zhang (2001, 2002)'s likelihood ratio tests under the null of exponentiality, employing response surface analysis—an approach not extensively explored in the existing literature.

Zhang (2001, 2002) derived goodness-of-fit tests based on the likelihood ratio for verifying whether a hypothesized distribution aligns with observed data. Monte Carlo studies proposed by Zhang and Wu (2005) and Torabi et al. (2016) showed that these tests outperform conventional normality tests, such as the Jarque-Bera (Jarque and Bera, 1987), Shapiro-Wilk (Shapiro and Wilk, 1965), and Anderson-Darling (Anderson and Darling, 1952, 1954) tests. Recently, Ossai et al. (2022) examined 91 tests for exponentiality and compared 40 of them via Monte Carlo simulations, showing that in many cases, Zhang (2001, 2002)'s likelihood ratio tests offer higher power than their competitors. A limitation, however, is that the asymptotic distributions of these tests remain unknown, and a small number of critical values were reported. Practitioners are forced to resort to Monte Carlo simulations to obtain critical values for unlisted sample sizes. Despite the superior performance of these tests in many scenarios, this requirement has constrained their adoption in practice.

Response surface methodology is often employed to estimate asymptotic distributions or asymptotic critical value functions for test statistics that do not follow standard distributions or deviate from their null asymptotic distributions. For example, Lawford (2005) and Wuertz and Katzgraber (2005) used this method to generate approximate critical value functions for variants of the Jarque-Bera test (Jarque and Bera, 1987; Urzua, 1996), correcting for their finite-sample properties. Similarly, MacKinnon (2010) relied on response surface analysis to obtain finite-sample critical values for Dickey-Fuller unit root (Dickey and Fuller, 1979; Fuller, 2009) and Engle-Granger cointegration tests (Engle and Granger, 1987), as neither test follows a standard asymptotic distribution. In the same vein, Munir et al. (2023) applied response surface analysis to compute approximate critical value functions for Zhang (2001, 2002)'s likelihood ratio tests, albeit under normality. While response surface analysis entails extensive Monte Carlo experiments, its utility is twofold. First, it can efficiently approximate critical values for tests with no closed-form limit distributions. Second, it condenses the outcomes from only a limited number of simulations into tables covering all sample sizes.

Building on this framework, the present study fills a gap in the literature by estimating asymptotic critical value functions for Zhang (2001, 2002)'s likelihood ratio tests under the null of the exponential distribution. These functions are derived by regressing Monte Carlo-based critical values on a small set of functions of the sample size, allowing practitioners to generate finite-sample critical values quickly, even with a standard calculator. We also introduce a methodological improvement by using median regression to estimate the response surfaces, thereby enhancing robustness and mitigating potential biases due to outliers or heteroskedasticity in the simulated data.

In what follows, Section 2 summarizes the formulation of Zhang (2001, 2002)'s likelihood ratio tests. In Section 3, we discuss the Monte Carlo setup, the response surface estimation, and evaluate the finite-sample performance of the proposed critical value functions. In Section 4, a real-data illustration is offered with three applications: inter-failure times of air conditioning systems, lifespans of ball bearings, and durations of blood cancer patients. Conclusions and potential avenues for further research are provided in Section 5.

2. Likelihood ratio-based tests

Zhang (2001, 2002) proposed a family of likelihood ratio-based goodness-of-fit tests to assess whether a given dataset follows an exponential distribution. These tests rely on the empirical distribution function and are based on three distinct test statistics: Z_K, Z_A, and Z_C, each capturing different aspects of the deviation between the observed and hypothesized distributions. In this section, we detail these tests.

2.1 *Test statistics*

Let X be a random variable with continuous cumulative distribution function $G(x)$, and let $X_1, \dots, X_T$ be a random sample of size T from X. Denote the order statistics by $X_{(1)}, \dots, X_{(T)}$. To test the null hypothesis H_0: $G(x) = G_0(x)$ against the alternative H_1: $G(x) \neq G_0(x)$, the likelihood ratio-based tests are defined using the empirical cumulative distribution function and the test statistics given by

$$Z_K = \max_{1 \le t \le T} \left\{ \left(t - \frac{1}{2}\right) \log\left(\frac{t-\frac{1}{2}}{TG_0(X_{(t)})}\right) + (T - t + \tfrac{1}{2}) \log\left(\frac{T-t+\frac{1}{2}}{T(1-G_0(X_{(t)}))}\right) \right\},$$

$$Z_A = -\sum_{t=1}^{T} \left\{ \frac{\log\{G_0(X_{(t)})\}}{T-t+\frac{1}{2}} + \frac{\log\{1-G_0(X_{(t)})\}}{t-\frac{1}{2}} \right\},$$

$$Z_C = \sum_{t=1}^{T} \left\{ \log\left(\frac{G_0(X_{(t)})^{-1}-1}{\left(\left(T-\frac{1}{2}\right)/\left(t-\frac{3}{4}\right)\right)-1}\right) \right\}^2,$$

where $G_0(X_{(t)})$ is the hypothesized cumulative distribution function evaluated at the tth order statistic.

The test statistic Z_K represents the maximum weighted logarithmic deviation between the empirical cumulative distribution function and $G_0(x)$, emphasizing the largest pointwise discrepancy. The statistic Z_A quantifies the sum of weighted logarithmic deviations, giving greater importance to observations in the tails. In contrast, Z_C is the sum of squared logarithmic deviations, making it particularly sensitive to multiple moderate departures from $G_0(x)$ rather than a few large ones. These tests are designed to detect both symmetric and asymmetric deviations from the hypothesized distribution and have been shown to be more powerful than classical tests such as Kolmogorov-Smirnov (Massey Jr , 1951; Stephens, 1970, 1974), Anderson-Darling (Anderson and Darling, 1952, 1954; Stephens, 1970, 1974), and Cramér-von Mises (Stephens, 1970, 1974) in various practical applications (Zhang, 2002). The null hypothesis is rejected for sufficiently large values of Z_K, Z_A, and Z_C.

2.2 *Distribution-free conditions and parameter estimation*

If $G_0(x)$ is fully specified, meaning that all parameters are known, the test statistics Z_K, Z_A, and Z_C are distribution-free, allowing their critical values to be determined directly through simulations under G_0. However, when $G_0(x)$ includes unknown parameters that need to be estimated from the sample, the distribution-free property no longer holds, and the test statistics do not follow standard tabulated distributions. In such cases, practitioners must rely on Monte Carlo simulations to obtain critical values for specific sample sizes T.

Zhang (2002) provided tabulated critical values for fully specified distributions, such as the exponential distribution with a known rate parameter and the normal distribution with a known mean and variance. However, these tables are only applicable when all distributional parameters are known in advance. When parameters must be estimated from the data, the critical values in these tables are no longer valid.

For instance, Zhang and Wu (2005) generated Monte Carlo-based critical values for a normal distribution in which the mean and variance were unknown. In such cases, the population mean μ and variance σ^2 must first be estimated using the sample mean and sample variance before computing the test statistics Z_K, Z_A, or Z_C. Because the estimation of parameters affects the distribution of these statistics, they become likelihood ratio tests for normality rather than direct goodness-of-fit tests.

More recently, Munir et al. (2023) applied response surface analysis to approximate these critical values as functions of the sample size T, enabling users to compute finite-sample critical values without requiring new simulations. This approach allows practitioners to obtain critical values efficiently using simple calculations rather than relying on precomputed tables limited to a fixed set of sample sizes.

2.3 *Application to the exponential distribution*

Drawing on the methodology of Munir et al. (2023), MacKinnon (2010), and Lawford (2005), the present work derives critical value functions for Z_K, Z_A, and Z_C under the assumption $G_0(x) = 1 - e^{-\lambda x}$ for $x \geq 0$, where λ is unknown. Similar to the normal case, we estimate λ by its sample counterpart, $\widehat{\lambda} = (1/\overline{X})$, where $\overline{X} = (1/T)\sum_{t=1}^{T} X_t$, and then substitute $\widehat{\lambda}$ into the formulas for Z_K, Z_A, and Z_C. This approach extends the applicability of the likelihood ratio tests to the exponential distribution with an unknown rate parameter. In the following sections, we develop convenient asymptotic critical value functions for these tests, enabling practitioners to conduct exponentiality checks without the need for case-by-case Monte Carlo simulations.

3. Monte Carlo experiments

This section describes the Monte Carlo simulations used to estimate the asymptotic critical value functions for the test statistics Z_K, Z_A, and Z_C. By systematically varying the sample size T and generating repeated realizations under the null hypothesis of exponentiality, response surface analysis is applied to model the relationship between simulated critical values and sample size, allowing for efficient approximation of finite-sample critical values.

3.1 *Simulation design*

In the present context, the response variable is the finite-sample critical value of each test statistic (Z_K, Z_A, and Z_C), and the control variable is the sample size T. The procedure for obtaining the response and control variables is as follows. Using simulation code and the `MonteCarlo` package (Leschinski, 2019) of the `R` software (R Core Team, 2024), 20,000 realizations of Z_K, Z_A, and Z_C were generated under the null hypothesis of exponentiality with $\lambda = 1$ for each integer T in the interval $[5, 4500]$. For each test statistic, the $10\%, 20\%, 5\%, 1\%, 2\%, 0.5\%, 2.5\%$, and 0.1% critical values (that is, quantiles) were computed as the $q(20{,}000)$th largest values, where $q \in \{90\%, 80\%, 95\%, 99\%, 98\%, 99.5\%, 97.5\%, 99.9\%\}$. Since there are 4496 integer values of T from 5 through 4500, this step yielded 4496 simulated critical values for each of the eight percentage points and for each of the three test statistics. Note that Z_K, Z_A, and Z_C are invariant under affine transformations of the data (Zhang and Wu, 2005), implying that the resulting critical values are also invariant to the specific choice of λ. Consequently, the estimated response surfaces are valid for all λ.

3.2 *Response surface specification*

To estimate critical values as smooth functions of T, a parsimonious functional form is specified in which the simulated critical values are regressed on an intercept and power transformations of $1/T$. This approach follows guidance from MacKinnon (2010), Munir et al. (2023), and Lawford (2005), who highlight the importance of carefully choosing a functional form. After extensive experimentation with alternatives, a specification proved both parsimonious and accurate is given by

$$cv^q(T)_l = \gamma_0 + \gamma_1 T^{-\frac{1}{2}} + \gamma_2 T^{-1} + \gamma_3 T^{-\frac{3}{2}} + \gamma_4 T^{-2} + \epsilon_t, \quad l \in \{Z_K, Z_A, Z_C\},$$

where γ_0 represents the asymptotic critical value since $T^{-\frac{1}{2}}$, T^{-1}, $T^{-\frac{3}{2}}$, and T^{-2} vanish as $T \to \infty$. The parameters $\gamma_1, \gamma_2, \gamma_3$, and γ_4 capture the shape of the finite-sample adjustment, while ϵ_t is the error term. The dependent variable $cv^q(T)_l$ is the simulated critical value of statistic l at percentage point q for sample size T.

3.3 *Median regression estimation*

Whereas earlier studies (MacKinnon, 2010; Lawford, 2005; Wuertz and Katzgraber, 2005; Kiefer and Vogelsang, 2005; Munir et al., 2023) commonly adopted ordinary least squares (OLS) or feasible generalized least squares (GLS) to estimate response surface regressions, the present study employs median regression, a special case of quantile regression, for two primary reasons. First, the goal is to predict the conditional median of the test statistics' finite-sample distributions, aligning naturally with median regression. Second, median regression is more robust to outliers and deviations from distributional assumptions such as homoscedasticity and normality (Francis and Nwakuya, 2022).

All median regressions were implemented via an `R` package named `quantreg` (Koenker, 2021), using the function `rq` with the Barrodale and Roberts (br) algorithm (Koenker and D'Orey, 1987; Koenker and D'Orey, 1994) for $\tau = 0.5$. Consequently, the final estimated response surface functions for each test statistic $l \in \{Z_K, Z_A, Z_C\}$ are stated as

$$\widehat{\mathrm{CV}}^q(T)_{Z_K} = \widehat{\gamma}_0 + \widehat{\gamma}_1 T^{-\frac{1}{2}} + \widehat{\gamma}_2 T^{-1} + \widehat{\gamma}_3 T^{-\frac{3}{2}} + \widehat{\gamma}_4 T^{-2}, \tag{3.1}$$

$$\widehat{\mathrm{CV}}^q(T)_{Z_A} = \widehat{\gamma}_0 + \widehat{\gamma}_1 T^{-\frac{1}{2}} + \widehat{\gamma}_2 T^{-1} + \widehat{\gamma}_3 T^{-\frac{3}{2}} + \widehat{\gamma}_4 T^{-2}, \tag{3.2}$$

$$\widehat{\mathrm{CV}}^q(T)_{Z_C} = \widehat{\gamma}_0 + \widehat{\gamma}_1 T^{-\frac{1}{2}} + \widehat{\gamma}_2 T^{-1} + \widehat{\gamma}_3 T^{-\frac{3}{2}} + \widehat{\gamma}_4 T^{-2}, \tag{3.3}$$

where $\widehat{\gamma}_0$, $\widehat{\gamma}_1$, $\widehat{\gamma}_2$, $\widehat{\gamma}_3$, and $\widehat{\gamma}_4$ are robust estimates of the corresponding parameters. This approach reduces sensitivity to outliers and heteroskedasticity, and the resulting response functions tend to predict critical values reliably across a wide range of sample sizes. The estimated coefficients for each statistic and quantile are displayed in Tables 1, 2, and 3. Except for $\widehat{\gamma}_4$ stated in Equation (3.3) at the 99.9% quantile, all coefficients are statistically significant at the 0.01 level.

The model presented in Equation (3.2) was selected based on low absolute median and mean deviations, parsimony, and a high pseudo-R^2. In all cases, at least 95% of the variability in the simulated data was explained. Introducing higher-order powers of $1/T$ provided negligible improvements, reinforcing the adequacy of the chosen specification.

3.4 *Illustrative examples of computed critical values*

The following examples demonstrate how to use the estimated response surface functions to compute approximate finite-sample critical values.

Example 3.1 For a sample size of $T = 20$ at the 5% significance level, the estimated critical value of Z_K is $\widehat{\text{CV}}^{0.95}(20)_{Z_K} = 5.27666 - 35.70992 \cdot (20)^{-\frac{1}{2}} + 249.29744 \cdot (20)^{-1} - 873.27165 \cdot (20)^{-\frac{3}{2}} + 1039.49163 \cdot (20)^{-2} = 2.591806$.

Example 3.2 For a sample size of $T = 20$ at the 5% significance level, the estimated critical value of Z_A is $\widehat{\text{CV}}^{0.95}(20)_{Z_A} = 3.28877 + 0.15695 \cdot (20)^{-1/2} + 9.88485 \cdot (20)^{-1} - 23.36337 \cdot (20)^{-3/2} + 19.76272 \cdot (20)^{-2} = 3.606304$.

Example 3.3 For a sample size of $T = 20$ at the 5% significance level, the estimated critical value of Z_C is $\widehat{\text{CV}}^{0.95}(20)_{Z_C} = 37.81069 - 382.90809 \cdot (20)^{-1/2} + 3273.82666 \cdot (20)^{-1} - 12553.87431 \cdot (20)^{-3/2} + 16121.01923 \cdot (20)^{-2} = 15.82714$.

3.5 *Regression coefficients and empirical validation*

Tables 1, 2, and 3 present the estimated response surface regressions for the test statistics Z_K, Z_A, and Z_C, respectively, across a range of quantiles. The coefficients are reported alongside their standard errors (in parentheses), and statistical significance is indicated by stars. These estimates required considerable computational effort, with each table taking approximately 378 minutes to complete. This highlights the intensive nature of the Monte Carlo simulations necessary to generate critical values across a wide range of sample sizes. Despite this, the final regression models exhibit excellent fit, as demonstrated by measures such as absolute median deviation, absolute mean deviation, and pseudo-R^2.

Each realization in the Monte Carlo experiments was generated independently from an exponential distribution with parameter $\lambda = 1$. Consequently, every simulated critical value corresponds to a distinct sample of size T, ensuring that the critical values used in the response surface regressions are mutually independent. This independence validates the assumption that the residuals in the response surface regressions are not affected by correlation structures.

A key distinction between the present study and Munir et al. (2023) lies in the choice of distribution and estimation methodology. While Munir et al. (2023) developed asymptotic critical value functions for Z_K, Z_A, and Z_C under the normal distribution, employing feasible GLS for estimation, the present study derives these functions under the exponential distribution using median regression. This methodological choice ensures robustness to outliers and heteroskedasticity, potentially leading to more reliable finite-sample approximations of the critical values.

Similar response surface regression tables for Z_A and Z_C are provided in Tables 2 and 3, respectively. The estimates confirm that the response surface approach effectively captures the simulated quantiles of each test statistic. Notably, the parameters governing finite-sample corrections ($\widehat{\gamma}_1, \widehat{\gamma}_2, \widehat{\gamma}_3$, and $\widehat{\gamma}_4$) demonstrate consistent trends across the three test statistics.

Overall, the proposed regression models provide an efficient and accurate approximation of the critical values for the likelihood ratio-based goodness-of-fit tests for exponentiality. The response surface methodology, combined with median regression estimation, ensures reliable and computationally efficient critical values that can be applied across a wide range of sample sizes without requiring additional Monte Carlo simulations.

Table 1. Estimated response surface regressions for the Z_K test.

	Quantiles							
	0.90	0.80	0.95	0.99	0.98	0.995	0.975	0.999
$\widehat{\gamma}_0$	4.559***	3.819***	5.277***	6.890***	6.204***	7.580***	5.976***	9.125***
	(0.008)	(0.007)	(0.009)	(0.012)	(0.010)	(0.027)	(0.010)	(0.019)
$\widehat{\gamma}_1$	−33.644***	−31.329***	−35.710***	−38.690***	−37.835***	−39.883***	−37.139***	−40.545***
	(0.659)	(0.596)	(0.731)	(0.926)	(0.821)	(2.102)	(0.787)	(1.164)
$\widehat{\gamma}_2$	240.807***	234.276***	249.297***	247.484***	253.494***	250.316***	248.913***	224.604***
	(15.005)	(14.002)	(15.987)	(18.713)	(17.474)	(42.679)	(16.688)	(17.313)
$\widehat{\gamma}_3$	−857.881***	−865.486***	−873.272***	−809.500***	−857.791***	−815.743***	−844.014***	−668.524***
	(109.899)	(105.566)	(112.717)	(121.289)	(118.725)	(276.303)	(113.766)	(76.706)
$\widehat{\gamma}_4$	1,038.907***	1,096.235***	1,039.492***	908.635***	991.465***	914.389*	978.929***	702.599***
	(230.625)	(224.458)	(232.893)	(229.158)	(229.886)	(512.404)	(222.363)	(100.952)

where the standard errors are in parentheses. The estimated finite-sample critical value for Z_K is $\widehat{\text{CV}}^q(T)_{Z_K} = \widehat{\gamma}_0 + \widehat{\gamma}_1 T^{-\frac{1}{2}} + \widehat{\gamma}_2 T^{-1} + \widehat{\gamma}_3 T^{-\frac{3}{2}} + \widehat{\gamma}_4 T^{-2}$. Significance levels: $*$p-value < 0.1; $**$ p-value < 0.05; $***$p-value < 0.01.

Table 2. Estimated response surface regressions for the Z_A test.

	Quantiles							
	0.90	0.80	0.95	0.99	0.98	0.995	0.975	0.999
$\widehat{\gamma}_0$	3.289***	3.289***	3.289***	3.289***	3.289***	3.289***	3.289***	3.289***
	(0.00004)	(0.00005)	(0.00002)	(0.0001)	(0.0001)	(0.0002)	(0.0001)	(0.001)
$\widehat{\gamma}_1$	0.143***	0.124***	0.157***	0.182***	0.180***	0.175***	0.175***	0.153***
	(0.004)	(0.004)	(0.002)	(0.010)	(0.008)	(0.016)	(0.013)	(0.055)
$\widehat{\gamma}_2$	7.870***	5.980***	9.885***	15.041***	12.524***	17.844***	11.847***	24.920***
	(0.102)	(0.120)	(0.042)	(0.252)	(0.205)	(0.418)	(0.330)	(1.479)
$\widehat{\gamma}_3$	−20.055***	−17.118***	−23.363***	−31.233***	−25.898***	−37.157***	−25.132***	−55.149***
	(0.794)	(1.001)	(0.207)	(1.974)	(1.464)	(3.137)	(2.566)	(11.962)
$\widehat{\gamma}_4$	17.892***	16.328***	19.763***	20.877***	17.333***	23.888***	17.818***	38.502
	(1.660)	(2.221)	(0.310)	(3.717)	(2.753)	(5.967)	(5.171)	(24.080)

where the standard errors are in parentheses. The estimated finite-sample critical value for Z_A is $\widehat{\text{CV}}^q(T)_{Z_A} = \widehat{\gamma}_0 + \widehat{\gamma}_1 T^{-\frac{1}{2}} + \widehat{\gamma}_2 T^{-1} + \widehat{\gamma}_3 T^{-\frac{3}{2}} + \widehat{\gamma}_4 T^{-2}$. Significance levels: $*$p-value < 0.1; $**$ p-value < 0.05; $***$p-value < 0.01.

Table 3. Estimated response surface regressions for the Z_C test.

	Quantiles							
	0.90	0.80	0.95	0.99	0.98	0.995	0.975	0.999
$\widehat{\gamma}_0$	32.713***	27.492***	37.811***	49.906***	44.633***	55.407***	42.958***	70.460***
	(0.096)	(0.092)	(0.115)	(0.151)	(0.126)	(0.161)	(0.125)	(0.237)
$\widehat{\gamma}_1$	−354.440***	−318.990***	−382.908***	−421.130***	−411.907***	−419.738***	−406.623***	−403.581***
	(8.331)	(7.981)	(9.760)	(12.024)	(10.283)	(12.634)	(10.352)	(14.789)
$\widehat{\gamma}_2$	3,111.885***	2,851.427***	3,273.827***	3,326.343***	3,397.563***	3,154.997***	3,407.779***	2,805.663***
	(202.502)	(195.396)	(229.467)	(261.862)	(231.671)	(267.657)	(236.583)	(222.238)
$\widehat{\gamma}_3$	−12,180.820***	−11,247.450***	−12,553.870***	−12,030.550***	−12,632.960***	−11,036.950***	−12,954.710***	−9,319.434***
	(1,562.181)	(1,517.802)	(1,709.049)	(1,815.881)	(1,684.489)	(1,814.391)	(1,722.326)	(1,004.909)
$\widehat{\gamma}_4$	15,881.440***	14,726.180***	16,121.020***	14,499.770***	15,508.470***	12,994.700***	16,544.420***	10,737.880***
	(3,327.413)	(3,314.862)	(3,602.222)	(3,729.739)	(3,520.447)	(3,589.636)	(3,604.538)	(1,295.723)

where the standard errors are in parentheses. The estimated finite-sample critical value for Z_C is $\widehat{\text{CV}}^q(T)_{Z_C} = \widehat{\gamma}_0 + \widehat{\gamma}_1 T^{-\frac{1}{2}} + \widehat{\gamma}_2 T^{-1} + \widehat{\gamma}_3 T^{-\frac{3}{2}} + \widehat{\gamma}_4 T^{-2}$. Significance levels: $*$p-value < 0.1; $**$ p-value < 0.05; $***$p-value < 0.01.

3.6 *Finite-sample performance*

The finite-sample properties of the proposed response surface functions were evaluated by examining the empirical test sizes. Specifically, one would like the probability of rejecting H_0 when H_0 is true (that is, the empirical size) to closely match the nominal significance level α. To this end, 20,000 samples were generated under the exponential distribution with $\lambda = 1$ for the sample sizes: $T \in \{20, 25, 50, 75, 100, 150, 200, 350, 450, 550, 650, 750, 850, 950, 1050, 1150, 1250, 1350, 1450, 1550, 1700, 1850, 2000\}$. The test statistics Z_K, Z_A, and Z_C were computed for each sample, and the rejection frequencies were determined by comparing the computed test statistic to the estimated critical value $\widehat{\text{CV}}^q(T)_l$ at significance levels $\alpha \in \{0.001, 0.005, 0.01, 0.02, 0.025, 0.05, 0.1, 0.2\}$.

Tables 4, 5, and 6 summarize the empirical sizes of Z_K, Z_A, and Z_C, respectively. Each of these tables took roughly 30 minutes to generate under the `MonteCarlo` package (Leschinski, 2019) of `R`. The results confirm that for $T \geq 20$, the response surface estimates given in Equations (3.1), (3.2), and (3.3) yield rejection rates very close to the nominal significance levels for all tested quantiles. When $5 \leq T < 20$, users may directly consult the simulated critical values presented in Tables 10, 11, 12 in Appendix B.

Overall, the difference between estimated and simulated critical values is typically minimal. For instance, at $T = 20$ and $\alpha = 0.05$, the estimated critical values of Z_K, Z_A, and Z_C closely match the simulated ones, differing only in the fourth or fifth decimal place. Notably, a parsimonious median regression specification is sufficient to achieve these results, whereas OLS or GLS approaches in analogous settings often require including higher-order terms in $1/T$ (Lawford, 2005; Munir et al., 2023).

All computations were performed on a laptop with an AMD Ryzen 7 2700U processor (2.20 GHz) and 8GB RAM, running `R` version 3.6.2 (R Core Team, 2024) under Microsoft Windows 10 Home (version 22H2). Data frames were exported to Microsoft Excel 2010 via the `writexl` package (Ooms, 2021) and imported from Excel `.xls` files using `readxl` (Wickham and Bryan, 2019). Complete `R` scripts are available in Appendix A.

Table 4. Empirical size of the Z_K test using $\widehat{\text{CV}}^q(T)_{Z_K}$.

	α							
T	0.001	0.005	0.01	0.02	0.025	0.05	0.1	0.2
20	0.00105	0.00405	0.00855	0.01905	0.02445	0.04575	0.09435	0.19435
25	0.00135	0.00485	0.00945	0.01900	0.02520	0.04725	0.09955	0.20025
50	0.00130	0.00460	0.00995	0.02070	0.02385	0.04950	0.10310	0.20265
75	0.00090	0.00455	0.00940	0.02005	0.02540	0.04945	0.09780	0.20680
100	0.00065	0.00500	0.00935	0.02080	0.02215	0.05080	0.10015	0.19770
150	0.00135	0.00485	0.00870	0.02105	0.02690	0.05255	0.10090	0.19655
200	0.00095	0.00485	0.01090	0.02045	0.02340	0.04850	0.10105	0.20245
350	0.00095	0.00465	0.01050	0.02240	0.02615	0.05065	0.10045	0.20210
450	0.00135	0.00515	0.01065	0.02170	0.02695	0.04935	0.09980	0.20105
550	0.00110	0.00500	0.00995	0.02150	0.02475	0.05085	0.09990	0.20490
650	0.00120	0.00545	0.01070	0.02000	0.02465	0.04920	0.09840	0.20135
750	0.00110	0.00620	0.01015	0.01980	0.02565	0.05125	0.10340	0.19970
850	0.00120	0.00555	0.01085	0.01895	0.02720	0.05105	0.10410	0.20180
950	0.00085	0.00480	0.01035	0.01945	0.02470	0.05060	0.09940	0.19515
1050	0.00095	0.00435	0.00980	0.02140	0.02540	0.05055	0.09400	0.19050
1150	0.00105	0.00565	0.00980	0.01935	0.02475	0.05010	0.10215	0.19620
1250	0.00110	0.00590	0.01030	0.01940	0.02600	0.04790	0.10000	0.20030
1350	0.00075	0.00570	0.01070	0.02095	0.02830	0.05060	0.10065	0.19490
1450	0.00070	0.00500	0.01025	0.02080	0.02410	0.05250	0.10010	0.20335
1550	0.00110	0.00470	0.01030	0.01990	0.02415	0.04870	0.09535	0.20000
1700	0.00065	0.00510	0.01090	0.01875	0.02410	0.05410	0.09880	0.19515
1850	0.00090	0.00525	0.00995	0.02015	0.02515	0.05030	0.10350	0.19975
2000	0.00070	0.00490	0.01010	0.02110	0.02425	0.05215	0.09895	0.20670

where α is the significance level and T is the sample size.

Table 5. Empirical size of the Z_A test using $\widehat{\mathrm{CV}}^q(T)_{Z_A}$.

	α							
T	0.001	0.005	0.01	0.02	0.025	0.05	0.1	0.2
20	0.00090	0.00475	0.00975	0.01920	0.02380	0.05150	0.10205	0.19910
25	0.00090	0.00440	0.00930	0.01835	0.02605	0.05140	0.09790	0.20365
50	0.00125	0.00485	0.00945	0.01795	0.02415	0.04895	0.09915	0.19800
75	0.00105	0.00515	0.01015	0.01975	0.02480	0.04950	0.09430	0.19685
100	0.00095	0.00495	0.01035	0.01920	0.02550	0.05015	0.10360	0.19490
150	0.00105	0.00530	0.00970	0.02040	0.02615	0.05140	0.10325	0.20115
200	0.00165	0.00385	0.00960	0.01885	0.02525	0.04975	0.10220	0.20225
350	0.00135	0.00480	0.00985	0.02160	0.02525	0.05345	0.10265	0.20275
450	0.00135	0.00485	0.00975	0.02095	0.02515	0.05220	0.10390	0.20705
550	0.00090	0.00475	0.00925	0.02120	0.02460	0.05245	0.10265	0.20290
650	0.00160	0.00515	0.01045	0.02140	0.02390	0.05045	0.10205	0.20675
750	0.00105	0.00480	0.01090	0.01945	0.02560	0.04870	0.10305	0.19905
850	0.00140	0.00500	0.00995	0.02115	0.02505	0.05085	0.09980	0.20380
950	0.00075	0.00480	0.00895	0.01825	0.02670	0.04965	0.09910	0.20710
1050	0.00100	0.00480	0.01120	0.02045	0.02525	0.05160	0.10205	0.20000
1150	0.00105	0.00515	0.01015	0.02080	0.02545	0.05150	0.09940	0.19885
1250	0.00100	0.00560	0.00995	0.02150	0.02335	0.05060	0.09555	0.19920
1350	0.00130	0.00440	0.01105	0.01825	0.02225	0.05205	0.09750	0.19840
1450	0.00100	0.00495	0.00910	0.01965	0.02420	0.05055	0.09840	0.20225
1550	0.00170	0.00465	0.00885	0.01800	0.02475	0.04875	0.09820	0.19580
1700	0.00120	0.00400	0.01005	0.02030	0.02460	0.05025	0.09750	0.19590
1850	0.00080	0.00425	0.01085	0.02130	0.02410	0.04495	0.09585	0.19230
2000	0.00145	0.00510	0.00995	0.01980	0.02280	0.05245	0.10180	0.19430

where α is the significance level, and T is the sample size.

Table 6. Empirical size of the Z_C test using $\widehat{\mathrm{CV}}^q(T)_{Z_C}$.

	α							
T	0.001	0.005	0.01	0.02	0.025	0.05	0.1	0.2
20	0.00060	0.00450	0.00805	0.01955	0.02345	0.04825	0.10815	0.21635
25	0.00095	0.00430	0.00800	0.01455	0.01985	0.03840	0.08250	0.17210
50	0.00070	0.00455	0.00735	0.01680	0.02145	0.03855	0.07490	0.15725
75	0.00070	0.00445	0.01005	0.01920	0.02190	0.04505	0.08500	0.17730
100	0.00045	0.00520	0.01145	0.02090	0.02585	0.04825	0.09645	0.18835
150	0.00095	0.00510	0.01060	0.02170	0.02785	0.05265	0.10225	0.20120
200	0.00090	0.00525	0.01110	0.02080	0.02845	0.05195	0.10525	0.21410
350	0.00100	0.00455	0.01080	0.02025	0.02590	0.05425	0.10485	0.20915
450	0.00120	0.00465	0.01040	0.02160	0.02640	0.05065	0.10775	0.20350
550	0.00120	0.00490	0.01015	0.01960	0.02550	0.05405	0.10460	0.20480
650	0.00080	0.00520	0.01000	0.01935	0.02465	0.05050	0.10580	0.20320
750	0.00110	0.00515	0.00965	0.01910	0.02585	0.04970	0.10100	0.20030
850	0.00095	0.00425	0.01000	0.01930	0.02370	0.04855	0.10135	0.20120
950	0.00075	0.00480	0.00950	0.02080	0.02405	0.04885	0.09860	0.20100
1050	0.00075	0.00525	0.00970	0.02020	0.02520	0.04825	0.09745	0.19985
1150	0.00070	0.00445	0.00860	0.01845	0.02455	0.04920	0.09970	0.19360
1250	0.00105	0.00540	0.00980	0.02115	0.02465	0.05220	0.09680	0.19665
1350	0.00105	0.00445	0.00880	0.02045	0.02525	0.05065	0.09920	0.19380
1450	0.00095	0.00485	0.00850	0.01920	0.02295	0.04820	0.09555	0.19700
1550	0.00085	0.00575	0.01005	0.01890	0.02365	0.05070	0.09705	0.19625
1700	0.00110	0.00515	0.00930	0.01975	0.02370	0.05055	0.09750	0.19610
1850	0.00115	0.00465	0.01070	0.01860	0.02445	0.04775	0.09890	0.20010
2000	0.00085	0.00505	0.00925	0.01900	0.02545	0.04970	0.09845	0.19645

where α is the significance level, and T is the sample size.

4. Empirical illustration

This section applies the proposed critical value functions to real-world data, using three datasets obtained from Sreelakshmi et al. (2018). The first dataset, denoted as Y_1, consists of the intervals between successive failures of air conditioning systems in 7913 jet airplanes, measured in hours, with a total of 27 observations. The second dataset, Y_2, includes 23 observations representing the number of revolutions, measured in millions, until the failure of ball bearings in a controlled life test. The third dataset, Y_3, comprises 40 observations corresponding to the ordered lifespans, measured in days, of patients diagnosed with blood cancer. These datasets provide practical examples to evaluate the performance of the likelihood ratio-based goodness-of-fit tests under real-world conditions.

4.1 *Datasets*

The datasets include the following variables:

- Y_1 —Intervals between successive failures (measured in hours) of air conditioning systems in 7913 jet airplanes.
- Y_2 —Number of revolutions (measured in millions) until the failure of ball bearings in a controlled life test study.
- Y_3 —Ordered lifespans (measured in days) of patients diagnosed with blood cancer, collected from one of the Health Ministry hospitals in Saudi Arabia.

All datasets were obtained from Sreelakshmi et al. (2018) and serve as practical examples to evaluate the performance of the proposed goodness-of-fit tests under real-world conditions. Table 7 details the datasets.

Table 7. Real-world datasets: air conditioning system failures, ball bearing failures, and patient lifespans.

Variable	Data								
Y_1	1	4	11	16	18	18	18	24	31
	39	46	51	54	63	68	77	80	82
	97	106	111	141	142	163	191	206	216
Y_2	17.88	28.92	33.00	41.52	42.12	45.60	48.40	51.84	51.96
	54.12	55.56	67.80	68.64	68.64	68.88	84.12	93.12	98.64
	105.12	105.84	127.92	128.04	173.40				
Y_3	115	181	255	418	441	461	516	739	743
	789	807	865	924	983	1024	1062	1063	1165
	1191	1222	1222	1251	1277	1290	1357	1369	1408
	1455	1478	1549	1578	1578	1599	1603	1605	1696
	1735	1799	1815	1852					

where Y_1 represents the failure intervals of air conditioning systems in jet airplanes, Y_2 corresponds to the number of revolutions until failure in a ball bearing test, and Y_3 denotes the ordered lifespans of patients diagnosed with blood cancer.

4.2 *Method of analysis*

The goodness-of-fit tests applied in this study include both the likelihood ratio-based statistics Z_K, Z_A, and Z_C, as well as classical exponentiality tests proposed by Kochar (1985) (K_o), Gnedenko et al. (1969) (G), and Harris (1976) (H_a). A comprehensive discussion of these classical tests, along with their mathematical formulations, is available in Ossai et al. (2022). All tests are conducted at a significance level of 5% ($\alpha = 0.05$).

To implement these tests, the likelihood ratio-based statistics Z_K, Z_A, and Z_C are computed using an `R` package named `DistributionTest` (Ning Cui, 2020), specifically employing the functions `zk.test(···)`, `za.test(···)`, and `zc.test(···)`. The classical tests K_o, G, and H_a are executed using an `R` package named `exptest` (Pusev et al., 2013), through the functions `kochar.exp.test(···)`, `gnedenko.exp.test(···)`, and `harris.exp.test(···)`, respectively. To obtain simulated p-values, the parameters `simulate.p.value = TRUE` and `nrepl = 20000` are specified. The critical values for Z_K, Z_A, and Z_C at $\alpha = 0.05$ are determined from the estimated response surface functions given in Equations (3.1)-(3.3), using sample sizes $T = 27$ for Y_1, $T = 23$ for Y_2, and $T = 40$ for Y_3.

4.3 *Results for likelihood ratio-based tests*

Table 8 shows the computed values of Z_K, Z_A, and Z_C for each dataset, along with their respective 5% critical values from the fitted response surfaces. The null hypothesis of exponentiality is rejected if the observed statistic exceeds the corresponding critical value.

Table 8. Application of Z_K, Z_A, and Z_C to the three datasets.

Dataset	Observed statistic			5% critical value		
	Z_K	Z_A	Z_C	$\widehat{\text{CV}}^{0.95}(T)_{Z_K}$	$\widehat{\text{CV}}^{0.95}(T)_{Z_A}$	$\widehat{\text{CV}}^{0.95}(T)_{Z_C}$
Y_1	0.558	3.330	5.026	2.839	3.546	18.005
Y_2	4.728	4.146	32.102	2.718	3.577	16.972
Y_3	7.358	4.291	57.249	3.061	3.481	19.565

where the critical values come from $\widehat{\text{CV}}^q(T)_l = \widehat{\gamma}_0 + \widehat{\gamma}_1 T^{-\frac{1}{2}} + \widehat{\gamma}_2 T^{-1} + \widehat{\gamma}_3 T^{-\frac{3}{2}} + \widehat{\gamma}_4 T^{-2}$, using $T = 27$, $T = 23$, and $T = 40$ for Y_1, Y_2, and Y_3, respectively, and $q = 0.95$.

From Table 8, we see that for Y_1, none of Z_K, Z_A, or Z_C exceed their respective critical values. By contrast, Y_2 and Y_3 both yield test statistics larger than the corresponding critical values, leading to a rejection of the exponentiality hypothesis at the 5% level.

4.4 *Results for classical exponentiality tests*

Table 9 reports the values of K_o, G, and H_a for the same datasets, as well as the associated simulated p-values (p-value$_1$, p-value$_2$, pvalue$_3$, respectively). A test rejects exponentiality if its computed statistic is large relative to simulated null distributions, yielding a small p-value.

Table 9. Application of K_o, G, and H_a to the three datasets.

Dataset	K_o	G	H_a	p–value$_1$	p–value$_2$	p–value$_3$
Y_1	1.627	1.308	0.676	0.230	0.247	0.846
Y_2	3.393	3.127	2.150	0.001	0.004	0.035
Y_3	5.291	6.485	1.882	0.000	0.000	0.025

where p-value$_1$, p-value$_2$, and p-value$_3$ are the simulated p-values for K_o, G, and H_a, respectively.

4.5 *Interpretation of findings*

The results indicate that for the air conditioning systems dataset (Y_1), all six tests fail to reject the null hypothesis of exponentiality. This suggests that the failure times of the jet-airplane air conditioning units follow a constant failure rate, meaning that the risk of failure does not change over time. This finding supports the assumption that preventive maintenance can be scheduled at fixed intervals without the need for additional adjustments based on increasing or decreasing hazard rates.

In contrast, for the ball bearings dataset (Y_2) and the blood cancer patients dataset (Y_3), both the likelihood ratio-based tests (Z_K, Z_A, Z_C) and the classical exponentiality tests (K_o, G, H_a) reject the exponentiality assumption. This indicates that the failure rates in these cases are not constant but vary over time, suggesting that more advanced reliability or survival models should be considered to account for time-dependent failure dynamics. These findings align with the results reported in Sreelakshmi et al. (2018), confirming the reliability of the response surface functions. This approach can be applied to other datasets where testing the exponential distribution hypothesis is of interest, providing a practical alternative to pre-tabulated critical values that are limited to specific sample sizes.

5. Conclusions, limitations, and future research

Goodness-of-fit tests based on the likelihood ratio, such as those proposed by Zhang (2001, 2002), are widely used to assess whether observed data follow a specific probability distribution. A key limitation of these tests is the absence of known asymptotic distributions for their test statistics, requiring practitioners to rely on precomputed critical values obtained via Monte Carlo simulations. However, existing tables of critical values cover only a limited range of sample sizes, necessitating additional simulations for cases not explicitly tabulated.

This study addresses this limitation by developing asymptotic critical value functions for the likelihood ratio-based goodness-of-fit tests under the null hypothesis of an exponential distribution. Monte Carlo simulations were used to generate critical values for a wide range of sample sizes, which were then modeled as functions of T using response surface regressions estimated via median regression. This approach allows practitioners to compute finite-sample critical values efficiently, without requiring new simulations, even for sample sizes not previously tabulated. The Monte Carlo results confirm that the estimated critical value functions provide accurate test sizes across different sample sizes, ensuring reliable application in practice.

The proposed functions have potential applications in various fields where testing for exponentiality is relevant, including reliability engineering, survival analysis, manufacturing, hydrology, and finance. In reliability engineering, these functions enable the assessment of whether a component's failure times follow an exponential distribution, which is a fundamental assumption in many system reliability models. In medical research, likelihood ratio-based goodness-of-fit tests can be applied to study survival times, and in manufacturing, they provide a means of verifying the consistency of product lifetimes with exponential failure models.

Despite the advantages of this approach, some limitations should be acknowledged. First, the response surface models developed here are specific to the test statistics Z_K, Z_A, and Z_C under the assumption of an exponential null hypothesis. Extending this methodology to other distributions or goodness-of-fit tests would require additional Monte Carlo simulations and model fitting. Second, while median regression was chosen for its robustness to outliers and heteroskedasticity, alternative regression techniques, such as generalized least squares (GLS), could be explored to assess whether further improvements in accuracy are possible. Additionally, the accuracy of the estimated response surface functions for very small sample sizes ($T < 20$) remains a potential limitation, as the variability in simulated critical values increases for small T.

Future research could focus on extending this methodology to other distributions, particularly in contexts where alternative parametric models are commonly used. Another avenue for further study involves evaluating the comparative performance of median regression, OLS, and GLS in estimating response surface models for critical values. Additionally, investigating the impact of different sample size ranges on the stability and predictive accuracy of the response surface functions could further refine their applicability.

Overall, this study provides a practical and computationally efficient solution to the problem of determining critical values for likelihood ratio-based goodness-of-fit tests under exponentiality. By enabling the direct computation of finite-sample critical values for a broad range of sample sizes, the proposed approach enhances the accessibility and applicability of these tests across multiple disciplines.

Appendix A: R codes for simulations and numerical analysis

This appendix contains the R scripts used to generate the simulated critical values, estimate the finite-sample sizes of the tests, and perform the numerical illustration. All simulations and computations were conducted on a laptop equipped with an AMD Ryzen 7 2700U 2.20 GHz processor and 8GB of RAM, running R version 3.6.2 (R Core Team, 2024) under Microsoft Windows 10 Home (version 22H2). The `writexl` package (Ooms, 2021) was used to export results to Excel, whereas `readxl` (Wickham and Bryan, 2019) was used to import Excel `.xls` files.

A.1. Codes for generating simulated critical values

Code for the test statistic Z_A

```
        library(MonteCarlo)
        library(writexl)
        library(PoweR)
        library(robustbase)
        library(DistributionTest)
        ###############################################
shahzad <- function(n, loc){
                x <- rexp(n, loc)
                n1 <- length(x)
                Data <- sort(x)
                MU <- mean(Data)
                F <- pexp(Data, 1/MU)
                i <- 1:n1
                part1 <- (log(F)) / (n1 - i + 0.5)
                part2 <- (log(1 - F)) / (i - 0.5)
                Z_A_Stat <- -sum(part1 + part2)
                return(list("Z_A_Stat" = Z_A_Stat))
        }
        ns <- seq(5, 4500, 1)
        loc_grid <- 1
        param_list <- list("n" = ns, "loc" = loc_grid)
s <- MonteCarlo(func = shahzad,
        nrep = 20000,
        param_list = param_list,
        ncpus = 1,
        time_n_test = FALSE)
f <- MakeFrame(s)
qaunt <- function(x){
quantile(x, probs = c(0.80, 0.90, 0.95, 0.98, 0.99, 0.995, 0.975, 0.999))}
qaunt.1 <- function(x){
quantile(x, probs = c(1 - 0.80, 1 - 0.90, 1 - 0.95, 1 - 0.98,
1 - 0.99, 1 - 0.995, 1 - 0.975, 1 - 0.999))}
uperlimit <- aggregate(abs(f$Z_A_Stat), list(f$n), qaunt)
lowerlimit <- aggregate(abs(f$Z_A_Stat), list(f$n), qaunt.1)
uper <- data.frame(n = uperlimit$Group.1, uperlimit$x)
lower <- data.frame(n = lowerlimit$Group.1, lowerlimit$x)
```

Code for the test statistic Z_C

```
        library(MonteCarlo)
        library(writexl)
        library(PoweR)
        library(robustbase)
        library(DistributionTest)
        ####################################################
        shahzad <- function(n, loc){
                x <- rexp(n, loc)
                n1 <- length(x)
                Data <- sort(x)
                MU <- mean(Data)
                F <- pexp(Data, 1/MU)
                i <- 1:n1
                F_Inv <- 1 / F
                Uper <- F_Inv - 1
                Lower <- ((n1 - 0.5) / (i - 0.75)) - 1
                lg <- (log(Uper / Lower))^2
                Z_C_Stat <- sum(lg)
                return(list("Z_C_Stat" = Z_C_Stat))
        }
        ns <- seq(5, 100, 1)
        loc_grid <- 1
        param_list <- list("n" = ns, "loc" = loc_grid)
s <- MonteCarlo(func = shahzad,
nrep = 20000,
param_list = param_list,
ncpus = 1,
time_n_test = FALSE)
f <- MakeFrame(s)
qaunt <- function(x){
quantile(x, probs = c(0.80, 0.90, 0.95, 0.98, 0.99, 0.995, 0.975, 0.999))}
qaunt.1 <- function(x){
        quantile(x, probs = c(1 - 0.80, 1 - 0.90, 1 - 0.95, 1 - 0.98,
1 - 0.99, 1 - 0.995, 1 - 0.975, 1 - 0.999))}
        uperlimit <- aggregate(abs(f$Z_C_Stat), list(f$n), qaunt)
        lowerlimit <- aggregate(abs(f$Z_C_Stat), list(f$n), qaunt.1)
        uper <- data.frame(n = uperlimit$Group.1, uperlimit$x)
        lower <- data.frame(n = lowerlimit$Group.1, lowerlimit$x)
```

Code for the test statistic Z_K

```
        library(MonteCarlo)
        library(writexl)
        library(PoweR)
        library(robustbase)
        library(DistributionTest)
        ####################################################
shahzad <- function(n, loc){
                x <- rexp(n, loc)
                n1 <- length(x)
                Data <- sort(x)
                MU <- mean(Data)
                F <- pexp(Data, 1/MU)
                i <- 1:n1
                part <- (i - 0.5)*log((i - 0.5) / (n1 * F)) +
                (n1 - i + 0.5)*log((n1 - i + 0.5) / (n1 * (1 - F)))
        Z_K_Stat <- max(part)
        # Jin Zhang (2005) Likelihood-ratio test for normality
        return(list("Z_K_Stat" = Z_K_Stat))}
```

```
        ns <- seq(2901, 3000, 1)
        loc_grid <- 1
        param_list <- list("n" = ns, "loc" = loc_grid)
        s <- MonteCarlo(func = shahzad,
        nrep = 20000,
        param_list = param_list,
        ncpus = 1,
        time_n_test = FALSE)
        f <- MakeFrame(s)
qaunt <- function(x){
quantile(x, probs = c(0.80, 0.90, 0.95, 0.98, 0.99, 0.995, 0.975, 0.999))}
qaunt.1 <- function(x){
        quantile(x, probs = c(1 - 0.80, 1 - 0.90, 1 - 0.95, 1 - 0.98,
        1 - 0.99, 1 - 0.995, 1 - 0.975, 1 - 0.999))}
        uperlimit <- aggregate(abs(f$Z_K_Stat), list(f$n), qaunt)
        lowerlimit <- aggregate(abs(f$Z_K_Stat), list(f$n), qaunt.1)
        uper <- data.frame(n = uperlimit$Group.1, uperlimit$x)
        lower <- data.frame(n = lowerlimit$Group.1, lowerlimit$x)
```

A.2. Codes for evaluating the size of the tests

Below are sample scripts demonstrating how we compute empirical sizes (rejection frequencies) for each test statistic at given significance levels and selected sample sizes.

Code for the test statistic Z_A

```
library(MonteCarlo)
library(writexl)
####################################################################
ZA_Crit <- function(n1, alpha){
if (alpha == 0.005){
        3.28869 + 0.17536*(1/sqrt(n1)) + 17.84430*(1/n1) -
        37.15707*(1/(n1^(1.5))) + 23.88810*(1/(n1^2))
                } else if (alpha == 0.01){
        3.28859 + 0.18186*(1/sqrt(n1)) + 15.04102*(1/n1) -
        31.23320*(1/(n1^(1.5))) + 20.87658*(1/(n1^2))
                } else if (alpha == 0.02){
        3.28859 + 0.17980*(1/sqrt(n1)) + 12.52359*(1/n1) -
        25.89824*(1/(n1^(1.5))) + 17.33339*(1/(n1^2))
                } else if (alpha == 0.05){
        3.28877 + 0.15695*(1/sqrt(n1)) + 9.88485*(1/n1) -
        23.36337*(1/(n1^(1.5))) + 19.76272*(1/(n1^2))
                } else if (alpha == 0.1){
        3.28887 + 0.14294*(1/sqrt(n1)) + 7.87019*(1/n1) -
        20.05474*(1/(n1^(1.5))) + 17.89241*(1/(n1^2))
                } else if (alpha == 0.2){
        3.28901 + 0.12433*(1/sqrt(n1)) + 5.97990*(1/n1) -
        17.11768*(1/(n1^(1.5))) + 16.32787*(1/(n1^2))
                } else if (alpha == 0.025){
        3.28863 + 0.17490*(1/sqrt(n1)) + 11.84741*(1/n1) -
        25.13250*(1/(n1^(1.5))) + 17.81844*(1/(n1^2))
                } else if (alpha == 0.001){
        3.28893 + 0.15318*(1/sqrt(n1)) + 24.91970*(1/n1) -
        55.14853*(1/(n1^(1.5))) + 38.50166*(1/(n1^2))
                }}
shahzad <- function(n, alpha, loc){
                x <- rexp(n, loc)
                n1 <- length(x)
                Data <- sort(x)
```

```
                    MU <- mean(Data)
                    F <- pexp(Data, 1/MU)
                    i <- 1:n1
                    part1 <- log(F) / (n1 - i + 0.5)
                    part2 <- log(1 - F) / (i - 0.5)
                    Z_A_Stat <- -sum(part1 + part2)
                    crit_val <- ZA_Crit(n1, alpha)
                    reject <- (Z_A_Stat > crit_val)
                    return(list("Z_A_Stat" = reject))
          }
ns <- c(20,25,50,75,100,150,200,350,450,550,650,750,850,950,1050,
          1150,1250,1350,1450,1550,1700,1850,2000)
          loc_grid <- 1
          alpha <- c(0.005, 0.01, 0.02, 0.05, 0.1, 0.2, 0.025, 0.001)
          param_list <- list("n" = ns, "alpha" = alpha, "loc" = loc_grid)
s <- MonteCarlo(func = shahzad,
          nrep = 20000,
          param_list = param_list,
          ncpus = 1,
          time_n_test = FALSE)
f <- MakeFrame(s)
          Size_ZA <- aggregate(f$Z_A_Stat, list(f$n, f$alpha), mean)
          Size_ZA
```

Code for the test statistic Z_K

```
library(writexl)
library(MonteCarlo)
###################################################################
ZK_Crit <- function(n1, alpha){
                    if (alpha == 0.005){
7.57973 - 39.88321*(1/sqrt(n1)) + 250.31620*(1/n1) -
815.74271*(1/(n1^(1.5))) + 914.38920*(1/(n1^2))
                    } else if (alpha == 0.01){
6.88999 - 38.69027*(1/sqrt(n1)) + 247.48406*(1/n1) -
809.49970*(1/(n1^(1.5))) + 908.63459*(1/(n1^2))
                    } else if (alpha == 0.02){
6.20409 - 37.83516*(1/sqrt(n1)) + 253.49402*(1/n1) -
857.79140*(1/(n1^(1.5))) + 991.46473*(1/(n1^2))
                    } else if (alpha == 0.05){
5.27666 - 35.70992*(1/sqrt(n1)) + 249.29744*(1/n1) -
873.27165*(1/(n1^(1.5))) + 1039.49163*(1/(n1^2))
                    } else if (alpha == 0.1){
4.55945 - 33.64369*(1/sqrt(n1)) + 240.80677*(1/n1) -
857.88121*(1/(n1^(1.5))) + 1038.90708*(1/(n1^2))
                    } else if (alpha == 0.2){
3.81926 - 31.32889*(1/sqrt(n1)) + 234.27607*(1/n1) -
865.48629*(1/(n1^(1.5))) + 1096.23525*(1/(n1^2))
                    } else if (alpha == 0.025){
5.97595 - 37.13854*(1/sqrt(n1)) + 248.91309*(1/n1) -
844.01383*(1/(n1^(1.5))) + 978.92859*(1/(n1^2))
                    } else if (alpha == 0.001){
9.12521 - 40.54450*(1/sqrt(n1)) + 224.60446*(1/n1) -
668.52370*(1/(n1^(1.5))) + 702.59870*(1/(n1^2))}}
shahzad <- function(n, alpha, loc){
                    x <- rexp(n, loc)
                    n1 <- length(x)
                    Data <- sort(x)
                    MU <- mean(Data)
                    F <- pexp(Data, 1/MU)
                    i <- 1:n1
```

```
                part <- (i - 0.5)*log((i - 0.5)/(n1*F)) +
                (n1 - i + 0.5)*log((n1 - i + 0.5)/(n1*(1 - F)))
                Z_K_Stat <- max(part)
                crit_val <- ZK_Crit(n1, alpha)
                reject <- (Z_K_Stat > crit_val)
                return(list("Z_K_Stat" = reject))}
ns <- c(20,25,50,75,100,150,200,350,450,550,650,750,850,950,1050,
1150,1250,1350,1450,1550,1700,1850,2000)
        loc_grid <- 1
        alpha <- c(0.005, 0.01, 0.02, 0.05, 0.1, 0.2, 0.025, 0.001)
        param_list <- list("n" = ns, "alpha" = alpha, "loc" = loc_grid)
s <- MonteCarlo(func = shahzad,
        nrep = 20000,
        param_list = param_list,
        ncpus = 1,
        time_n_test = FALSE)
        f <- MakeFrame(s)
        Size_Z_K <- aggregate(f$Z_K_Stat, list(f$n, f$alpha), mean)
        Size_Z_K
```

Code for the test statistic Z_C

```
library(writexl)
library(MonteCarlo)
#####################################################################
ZC_Crit <- function(n1, alpha){
        if (alpha == 0.005){
        55.40744 - 419.73794*(1/sqrt(n1)) + 3154.99716*(1/n1) -
        11036.95244*(1/(n1^(1.5))) + 12994.69580*(1/(n1^2))
} else if (alpha == 0.01){
        49.90601 - 421.12995*(1/sqrt(n1)) + 3326.34263*(1/n1) -
        12030.54622*(1/(n1^(1.5))) + 14499.77077*(1/(n1^2))
        } else if (alpha == 0.02){
        44.63341 - 411.90735*(1/sqrt(n1)) + 3397.56252*(1/n1) -
        12632.95936*(1/(n1^(1.5))) + 15508.46723*(1/(n1^2))
        } else if (alpha == 0.05){
        37.81069 - 382.90809*(1/sqrt(n1)) + 3273.82666*(1/n1) -
        12553.87431*(1/(n1^(1.5))) + 16121.01923*(1/(n1^2))
        } else if (alpha == 0.1){
        32.71291 - 354.43965*(1/sqrt(n1)) + 3111.88548*(1/n1) -
        12180.81877*(1/(n1^(1.5))) + 15881.44210*(1/(n1^2))
        } else if (alpha == 0.2){
        27.49222 - 318.98957*(1/sqrt(n1)) + 2851.42741*(1/n1) -
        11247.45314*(1/(n1^(1.5))) + 14726.17652*(1/(n1^2))
} else if (alpha == 0.025){
        42.95781 - 406.62265*(1/sqrt(n1)) + 3407.77939*(1/n1) -
        12954.71437*(1/(n1^(1.5))) + 16544.41711*(1/(n1^2))
} else if (alpha == 0.001){
        70.46000 - 403.58115*(1/sqrt(n1)) + 2805.66309*(1/n1) -
        9319.43436*(1/(n1^(1.5))) + 10737.88076*(1/(n1^2))}}
        shahzad <- function(n, alpha, loc){
                x <- rexp(n, loc)
                n1 <- length(x)
                Data <- sort(x)
                MU <- mean(Data)
                F <- pexp(Data, 1/MU)
                i <- 1:n1
                F_Inv <- 1 / F
                Uper <- F_Inv - 1
                Lower <- ((n1 - 0.5)/(i - 0.75)) - 1
                lg <- (log(Uper/Lower))^2
```

```
            Z_C_Stat <- sum(lg)
            crit_val <- ZC_Crit(n1, alpha)
            reject <- (Z_C_Stat > crit_val)
            return(list("Z_C_Stat" = reject))}
ns <- c(20,25,50,75,100,150,200,350,450,550,650,750,850,950,1050,
1150,1250,1350,1450,1550,1700,1850,2000)
loc_grid <- 1
alpha <- c(0.005, 0.01, 0.02, 0.05, 0.1, 0.2, 0.025, 0.001)
param_list <- list("n" = ns, "alpha" = alpha, "loc" = loc_grid)
s <- MonteCarlo(func = shahzad,
nrep = 20000,
param_list = param_list,
ncpus = 1,
time_n_test = FALSE)
f <- MakeFrame(s)
Size_ZC <- aggregate(f$Z_C_Stat, list(f$n, f$alpha), mean)
Size_ZC
```

A.3. Code for numerical illustration

The following code snippet illustrates how to conduct the empirical analysis discussed in the paper. It imports three datasets (Y_1, Y_2, and Y_3) from an Excel file and then computes the test statistics $\{Z_K, Z_A, Z_C\}$ using the `DistributionTest` package, as well as the classical exponentiality tests $\{K_o, G, H_a\}$ from `exptest`.

```
rm(list=ls())
library(readxl)
library(DistributionTest) # Jin Zhang (2005) Likelihood-ratio tests
library(exptest) # Kochar (1985) exponentiality tests
    ########################################
    # Load data from Excel files
    ########################################
    AirPlane <- read_excel("C:/Users/hp/Desktop/AirPlane.xlsx")
    attach(AirPlane)
    Y_1 <- c(na.omit(AirPlane$`Arrival Time`))
    Y_2 <- c(na.omit(AirPlane$`Ball Bearing`))
    Y_3 <- c(na.omit(AirPlane$`Blood Cancer`))
    ########################################
    # Compute Z_K, Z_A, Z_C statistics
    ########################################
    Z_K_test_Y1 <- zk.test(Y_1, "exp", N=0)
    Z_K_test_Y2 <- zk.test(Y_2, "exp", N=0)
    Z_K_test_Y3 <- zk.test(Y_3, "exp", N=0)
    Z_A_test_Y1 <- za.test(Y_1, "exp", N=0)
    Z_A_test_Y2 <- za.test(Y_2, "exp", N=0)
    Z_A_test_Y3 <- za.test(Y_3, "exp", N=0)
    Z_C_test_Y1 <- zc.test(Y_1, "exp", N=0)
    Z_C_test_Y2 <- zc.test(Y_2, "exp", N=0)
    Z_C_test_Y3 <- zc.test(Y_3, "exp", N=0)
    ########################################
    # Compute K_o, G, H_a
    ########################################
    K_o_test_Y1 <- kochar.exp.test(Y_1)
    K_o_test_Y2 <- kochar.exp.test(Y_2)
    K_o_test_Y3 <- kochar.exp.test(Y_3)
    gnedenko.exp.test(Y_1, simulate.p.value=TRUE, nrepl=20000)
    gnedenko.exp.test(Y_2, simulate.p.value=TRUE, nrepl=20000)
    gnedenko.exp.test(Y_3, simulate.p.value=TRUE, nrepl=20000)
    harris.exp.test(Y_1, simulate.p.value=TRUE, nrepl=20000)
    harris.exp.test(Y_2, simulate.p.value=TRUE, nrepl=20000)
    harris.exp.test(Y_3, simulate.p.value=TRUE, nrepl=20000)
```

```
##########################################
# View results
##########################################
Z_K_test_Y1$statistic
Z_K_test_Y2$statistic
Z_K_test_Y3$statistic
Z_A_test_Y1$statistic
Z_A_test_Y2$statistic
Z_A_test_Y3$statistic
Z_C_test_Y1$statistic
Z_C_test_Y2$statistic
Z_C_test_Y3$statistic
K_o_test_Y1$statistic
K_o_test_Y2$statistic
K_o_test_Y3$statistic
```

In the above scripts, certain paths (for example, at the link:
`"C:/Users/hp/Desktop/AirPlane.xlsx"`)
are user-specific and may need to be changed based on the local machine setup.

Appendix B: Simulated critical values for small sample sizes

This appendix presents precomputed simulated critical values (quantiles) for the test statistics Z_K, Z_A, and Z_C under the null hypothesis of exponentiality for small sample sizes ($5 \le T < 20$). These values allow practitioners to apply the goodness-of-fit tests without requiring additional Monte Carlo simulations when working with limited data.

Table 10. Simulated critical values or quantiles of Z_K for testing Exponentiality when the sample size is $5 \le T < 20$.

T	0.8	0.9	0.95	0.98	0.99	0.995	0.975	0.999
5	0.9029	1.2104	1.5375	1.9585	2.2979	2.6712	1.8646	3.7251
6	0.9950	1.3516	1.7039	2.1823	2.5025	2.9123	2.0643	4.0364
7	1.0863	1.4705	1.8488	2.3402	2.7444	3.2338	2.2100	4.5562
8	1.1491	1.5345	1.9306	2.4655	2.8917	3.3319	2.3284	4.4273
9	1.2132	1.6252	2.0486	2.6252	3.0279	3.4260	2.4632	4.5451
10	1.2493	1.6697	2.0940	2.6215	3.0647	3.5243	2.4961	4.5733
11	1.3007	1.7286	2.1931	2.8128	3.3146	3.8701	2.6679	5.1067
12	1.3704	1.8053	2.2333	2.8573	3.3211	3.7815	2.7211	4.9350
13	1.3739	1.8324	2.2809	2.8911	3.3871	3.8850	2.7334	5.1229
14	1.4208	1.8715	2.3043	2.9473	3.3721	3.7662	2.7704	5.0558
15	1.4619	1.9225	2.3975	3.0520	3.5675	4.0274	2.8998	5.2683
16	1.4993	1.9733	2.4487	3.0768	3.5861	4.0272	2.9405	5.0882
17	1.5285	1.9954	2.4660	3.1319	3.7105	4.2464	2.9564	5.5252
18	1.5358	2.0091	2.5168	3.2142	3.7307	4.2260	3.0392	5.2007
19	1.5804	2.0660	2.5403	3.1348	3.6465	4.1310	2.9998	5.4378

where these simulated critical values or quantiles provide very accurate empirical rates for Z_K when $5 \le T < 20$.

Table 11. Simulated critical values or quantiles of Z_A for testing Exponentiality whem the sample size is $5 \leq T < 20$.

T	0.80	0.90	0.95	0.98	0.99	0.995	0.975	0.999
5	3.6440	3.8253	3.9968	4.2139	4.3638	4.5681	4.1586	4.9488
6	3.6253	3.7914	3.9596	4.1686	4.3244	4.4827	4.1195	4.8682
7	3.6079	3.7541	3.9155	4.1162	4.2525	4.4030	4.0687	4.7448
8	3.5827	3.7194	3.8621	4.0488	4.1875	4.2884	4.0062	4.5785
9	3.5699	3.6956	3.8245	3.9963	4.1253	4.2486	3.9577	4.4886
10	3.5497	3.6672	3.7922	3.9510	4.0640	4.1743	3.9148	4.4503
11	3.5401	3.6489	3.7598	3.9147	4.0245	4.1519	3.8778	4.4060
12	3.5303	3.6294	3.7373	3.8859	4.0059	4.1189	3.8480	4.3701
13	3.5119	3.6077	3.7060	3.8433	3.9494	4.0554	3.8114	4.3195
14	3.5065	3.5984	3.6916	3.8118	3.9077	4.0208	3.7854	4.2183
15	3.4981	3.5842	3.6728	3.8012	3.8907	4.0023	3.7691	4.2476
16	3.4922	3.5731	3.6586	3.7853	3.8873	3.9714	3.7561	4.1892
17	3.4847	3.5657	3.6484	3.7626	3.8554	3.9300	3.7332	4.1929
18	3.4748	3.5522	3.6296	3.7440	3.8319	3.9143	3.7152	4.1014
19	3.4720	3.5452	3.6223	3.7236	3.8096	3.8980	3.6957	4.0467

where these simulated critical values or quantiles provide very accurate empirical rates for Z_A when $5 \leq T < 20$.

Table 12. Simulated critical values or quantiles of Z_C for testing exponentiality when the sample size is $5 \leq T < 20$.

T	0.80	0.90	0.95	0.98	0.99	0.995	0.975	0.999
5	6.5218	8.5145	10.3657	12.5005	14.7411	19.1350	12.0323	36.7313
6	6.9961	9.2152	11.2679	13.9601	16.5673	19.8773	13.2911	37.4765
7	7.5011	9.7111	11.9295	14.9112	18.0452	21.4916	14.2469	39.1280
8	7.7144	9.9935	12.3550	15.4851	18.2275	22.3413	14.6886	35.0512
9	8.0488	10.4532	12.7903	16.0254	18.4639	22.7810	15.2208	38.0523
10	8.1994	10.6612	13.1388	16.3769	19.0623	22.9626	15.6111	36.6808
11	8.5523	11.0301	13.5609	17.2958	20.7229	26.0848	16.2964	43.1090
12	8.7918	11.2908	13.8383	17.6049	20.8465	24.6428	16.7874	38.1393
13	8.8113	11.2947	13.9319	17.7608	20.9615	24.7130	16.7009	38.3636
14	9.1279	11.7106	14.3238	17.9471	21.1990	25.4752	16.9936	42.1110
15	9.2917	11.9598	14.5510	18.5562	21.4525	25.2264	17.5529	40.0107
16	9.5328	12.2342	14.9822	19.2337	22.4595	26.3605	18.0600	39.7224
17	9.6434	12.4683	15.2288	18.9524	22.4190	26.6964	18.1316	46.0979
18	9.6782	12.4891	15.4052	19.1593	22.6591	26.5489	18.2411	39.3925
19	9.9216	12.8402	15.6584	19.8835	23.2966	27.1591	18.5562	42.6114

where these simulated critical values or quantiles provide very accurate empirical rates for Z_C when $5 \leq T < 20$.

Statements

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Author contribution

Conceptualization, S.M.; formal analysis, S.M.; investigation, S.M.; methodology, S.M.; software, S.M.; validation, S.M.; visualization, S.M.; writing—original draft preparation, S.M.; writing—review and editing, S.M. The author has read and agreed to the published version of the article.

Conflicts of interest

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