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Statistical Modeling
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Statistical inference for first-order periodic autoregressive conditional heteroscedasticity models

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Abstract

This article investigates the statistical and asymptotic properties of first-order periodic autoregressive conditional heteroscedasticity (P-ARCH(1)) models, emphasizing both theoretical developments and empirical insights. The study introduces the class of P-ARCH models, highlighting how periodicity allows for distinct volatility dynamics across different seasons or regimes. By employing a vector autoregressive representation of the squared process, key probabilistic properties are derived under periodic stationarity conditions. A moment-based estimation approach using periodic Yule-Walker equations is proposed, and its consistency and asymptotic normality are established. Monte Carlo simulations assess the finite-sample performance of these estimators, particularly in comparison with least squares methods, and provide practical guidelines for implementation. The methodology is further illustrated through an empirical application to monthly log-stock returns of Intel Corporation, demonstrating that periodic ARCH models effectively capture time-varying heteroscedasticity driven by recurring seasonal or cyclical factors, outperforming approaches that assume constant volatility parameters. This study underscores the flexibility of P-ARCH(1) models in financial and economic applications where volatility patterns repeat over known intervals and highlights the advantages of a moment-based estimation strategy as a computationally efficient alternative to quasi-maximum likelihood methods. The findings provide clear directions for future research in extending the periodic ARCH framework to accommodate more complex dynamics and additional stylized features of real-world time series.

Keywords: Asymptotic normality · Consistency · Periodic ARCH model · Periodic Yule-Walker estimator · Periodically correlated processes · Statistical inference

Mathematics Subject Classification: Primary 62M05 · Secondary 62M10

1. Introduction

Financial time series often exhibit volatility clustering, wherein periods of high and low variability alternate over time. This phenomenon has motivated the development of econometric models that explicitly capture time-varying conditional heteroscedasticity, notably the autoregressive conditional heteroscedasticity (ARCH) model introduced by Engle (1982) and its extension to generalized ARCH (GARCH) proposed by Bollerslev (1986).

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Traditional ARCH and GARCH models assume constant parameters over time, but empirical studies, such as those by Franses (1998) and Hurd and Miamee (2007), have shown that financial time series often follow periodic or seasonal volatility patterns. Modeling such periodicity is crucial for analyzing financial returns, macroeconomic indicators, and energy markets, where seasonality is an intrinsic characteristic of the data (Ziel, 2015; Tsay, 2002).

To address these periodic characteristics, Bollerslev and Ghysels (1996) introduced periodic ARCH (P-ARCH) models as an extension of traditional heteroscedastic frameworks, allowing the conditional variance to vary periodically over time. Basawa and Lund (2001) further explored their properties, demonstrating their effectiveness in capturing systematic fluctuations in volatility due to seasonal effects, business cycles, and institutional trading patterns. Subsequent generalizations, such as periodic GARCH (P-GARCH) models developed by Aknouche and Bibi (2009) and further analyzed by Francq and Zakoian (2019), incorporate moving-average structures in volatility, so improving the modeling of long-range dependence in financial data.

Despite the growing use of periodic models, statistical inference for P-ARCH processes remains challenging. Estimating model parameters requires methods that account for periodicity while still ensuring statistical consistency and efficiency. Aknouche et al. (2018) and Aknouche and Al-Eid (2012) explored quasi-maximum likelihood estimation (QMLE) for periodic models, extending applicability beyond the standard GARCH setting. However, as noted by Bibi and Lescheb (2013) and Lund and Basawa (1999), the direct application of QMLE to periodic ARCH-type models requires additional considerations due to the time-dependent variance structure.

In this article, we focus on first-order PARCH —P-ARCH(1)— models and investigate the asymptotic properties of a moment-based estimation procedure using periodic Yule-Walker (YW) equations. Such an approach provides a computationally efficient alternative to QMLE and least squares (LS) estimation, making it appealing for large datasets. Specifically, we establish the consistency and asymptotic normality of these estimators, demonstrating their practical applicability. We also evaluate finite-sample performance through Monte Carlo simulations, comparing our proposed estimators with LS methods. Then, the methodology is applied to monthly log-stock returns of Intel Corporation (a dataset previously analyzed by Tsay (2002)), showing that periodic modeling effectively captures time-varying heteroscedasticity.

The main contributions of this study can be summarized as follows. First, the periodic YW estimation procedure for P-ARCH(1) models is formalized, and its consistency and asymptotic normality are established. Additionally, sufficient conditions for periodic stationarity are derived, ensuring the validity of the proposed estimation approach. To assess the performance of these estimators, Monte Carlo simulations are conducted to evaluate their finite-sample properties in comparison with alternative estimation techniques. At last, the methodology is applied to financial time series, demonstrating its advantages in capturing periodic volatility patterns.

This article is structured as follows. After the introduction, Section 2 provides a detailed presentation of the periodic ARCH model, including its vector representation, stationarity conditions, and higher-order moment properties. In Section 3, we explore the moment properties of the squared process and develop the YW estimation procedure, deriving its asymptotic properties and introducing a Wald test for periodicity. In Section 4, the performance of the proposed estimator is evaluated through Monte Carlo simulations and applies the methodology to real financial data. Section 5 presents concluding remarks and outlines directions for future research. Proofs of technical results are provided in Appendix.

2. Periodic ARCH model

This section introduces the P-ARCH model, which extends the standard ARCH framework by allowing time-varying volatility patterns that repeat over known periods. The mathematical formulation and key properties of the model are presented, including its vector representation, stationarity conditions, and higher-order moment properties.

2.1 *Definition and periodic version*

The concept of periodicity applies to both ARCH and GARCH models, collectively referred to as P-GARCH. These formulations are particularly relevant in applications where seasonal or cyclical volatility patterns have been observed (Bollerslev and Ghysels, 1996; Franses, 1998; Hurd and Miamee, 2007; Francq and Zakoian, 2019).

This article focuses on the first-order periodic ARCH (P-ARCH(1)) model, introduced by Bollerslev and Ghysels (1996) as an extension of the standard ARCH process proposed by Engle (1982). Unlike the traditional ARCH framework, which assumes constant parameters, P-ARCH incorporates periodic variation by allowing season-specific coefficients. This structure effectively captures recurring volatility fluctuations, making it particularly relevant for financial, macroeconomic, and energy market applications.

Formally, let $\{X_t, t \in \mathrm{Z}\}$ be a second-order process, meaning it has well-defined first and second moments (that is, finite mean and autocovariance function). Suppose this process has a known integer period $s \geq 1$, indicating that its statistical properties repeat every s time steps in a deterministic pattern. The P-ARCH(1) model extends the traditionalARCH(1) framework by allowing key parameters to vary across these s periods. Specifically, the P-ARCH(1) specification is given by

$$X_t = e_t h_t, \quad h_t^2 = \omega(t) + \alpha(t) X_{t-1}^2, \tag{2.1}$$

where $\{e_t\}$ is a sequence of independent and identically distributed (IID) random variables with zero mean, unit variance ($\kappa_1 = 1$), and finite higher moments $\kappa_k = \mathrm{E}[e_t^{2k}] < \infty$ for $k \geq 2$. The functions ω and α are periodic in t with period s, that is, $\omega(t + ks) = \omega(t) > 0$, $\alpha(t + ks) = \alpha(t) \geq 0$, for all $t, k \in \mathrm{Z}$. Thus, the conditional variance of X_t is given by $h_t^2 = \mathrm{Var}[X_t|\Im_{t-1}]$, where $\Im_{t-1}$ denotes all relevant information up to time $t-1$ (the σ-algebra generated by past observations). In a periodic framework of period s, each integer t can be expressed as $t = s\ell + \nu$, with $\nu \in \{1, \ldots, s\}$. Each value of ν is referred to as a season (or phase) within the period. Consequently, the functions $\omega(t)$ and $\alpha(t)$ remain constant for all t belonging to the same season ν.

The assumption $\omega(t) > 0$ for each t ensures strictly positive intercepts in every season, thus guaranteeing positivity of the conditional variance. Likewise, requiring $\alpha(t) \geq 0$ rules out negative coefficients, which would otherwise violate the ARCH structure.

To explicitly account for periodicity, we rewrite t as $t = s\ell + \upsilon$, where $\ell \in \mathrm{Z}$ and $\upsilon \in \{1, \ldots, s\}$. Define $X_t(\upsilon) = X_{s\ell+\upsilon}$, $h_t(\upsilon) = h_{s\ell+\upsilon}$, and $e_t(\upsilon) = e_{s\ell+\upsilon}$. Then, Equation (2.1) can be equivalently rewritten as

$$X_t(\upsilon) = e_t(\upsilon) h_t(\upsilon), \quad h_t^2(\upsilon) = \omega(\upsilon) + \alpha(\upsilon) X_t^2(\upsilon - 1), \quad \upsilon \in \{1, \ldots, s\}, \tag{2.2}$$

where $\omega(\upsilon)$ and $\alpha(\upsilon)$ remain constant within each season υ, enabling the model to capture season-specific or recurring changes in volatility over time.

2.2 *Periodic correlation and stationarity*

Under suitable conditions, a P-ARCH process belongs to the class of periodically correlated (PC) processes, which satisfy $\mathrm{E}[X_t] = \mathrm{E}[X_{t+s}]$, $\mathrm{Cov}[X_t, X_r] = \mathrm{Cov}[X_{t+s}, X_{r+s}]$, for all t, r. For $s = 1$, this notion coincides with conventional second-order stationarity. For $s > 1$, the process is globally nonstationary but remains stationary within each period (Franses, 1998; Hurd and Miamee, 2007; Francq and Zakoian, 2019).

Conditions for the existence of a strictly stationary (in the periodic sense) solution for a P-ARCH(1) model generally involve controlling the product of the ARCH coefficients or, equivalently, ensuring that the spectral radius of the associated companion matrix is strictly less than one. A sufficient condition is $\prod_{\upsilon=1}^{s} \alpha(\upsilon) < 1$, which guarantees the existence of a unique strictly periodic stationary and ergodic solution (Aknouche and Bentarzi, 2008; Lee and Shin, 2010; Sadoun and Bentarzi, 2022). For higher-order moments to be finite, a stronger restriction is required, such as $\prod_{\upsilon=1}^{s} \alpha(\upsilon)^2 < 1$, which ensures the feasibility of asymptotic normality results for moment-based estimators. These conditions are particularly relevant in financial applications, where volatility exhibits cyclical or seasonal patterns, such as monthly or weekly effects.

Several extensions of periodic ARCH and GARCH models have been proposed to capture different dynamics and enhance flexibility. For instance, Bibi and Lescheb (2010a) analyzed P-GARCH(p, q) processes, establishing consistency and asymptotic normality (CAN) under QMLE, while Bibi and Lescheb (2013) extended these results to P-ARMA models with P-GARCH errors. Aknouche et al. (2018) further developed a generalized QMLE approach applicable to a broader class of P-GARCH(p, q) processes.

To accommodate higher-order moments, Aknouche and Bentarzi (2008) introduced conditions for their existence in periodic GARCH models. Aliat and Hamdi (2019) explored Markov-switching versions, which integrate regime-switching with periodic coefficients. Transformations such as periodic log-GARCH were studied by Bibi and Hamdi (2024), while periodic exponential GARCH (P-EGARCH) models were introduced by Sadoun and Bentarzi (2022). Guerbyenne and Kessira (2019) proposed power periodic threshold GARCH models, establishing CAN for QMLE. Ziel (2015) examined multivariate P-GARCH settings, and Aknouche and Al-Eid (2012) investigated nonstationary periodic GARCH variants.

Methodologically, these contributions show the need to adapt traditional GARCH-type techniques to account for periodic or seasonal effects, whether through direct parameterization, regime-switching mechanisms, or specialized transformations. In the following sections, a moment-based estimation approach for the P-ARCH(1) model is developed, relying on YW equations to derive its asymptotic properties. This approach offers a computationally efficient alternative to QMLE, complementing the existing literature while maintaining well-defined asymptotics. A key aspect of the estimation procedure is the characterization of the squared process, which exhibits autocorrelation even when the original series $\{X_t\}$ itself appears uncorrelated. To analyze this structure, it is helpful to reformulate $\{X_t^2\}$ in a vector autoregressive form, allowing for a systematic derivation of moment-based estimators.

2.3 *Vector representation of the squared process*

Following the ideas presented in Gladyshev (1961), we start with the P-ARCH(1) equations $h_t^2(\upsilon) = \omega(\upsilon) + \alpha(\upsilon) X_{t-1}^2(\upsilon - 1)$, $X_t(\upsilon) = e_t(\upsilon) h_t(\upsilon)$, where $\upsilon \in \{1, \ldots, s\}$ denotes the season or period index. Squaring the second equation and substituting $h_t^2(\upsilon)$ from above yields the expression stated as

$$X_t^2(\upsilon) = \omega(\upsilon) + \alpha(\upsilon) X_{t-1}^2(\upsilon - 1) + \eta_t(\upsilon), \tag{2.3}$$

where $\eta_t(\upsilon) = X_t^2(\upsilon) - \mathrm{E}[X_t^2(\upsilon) | \Im_{t-1}] = h_t^2(\upsilon) \left(e_t^2(\upsilon) - 1\right)$ is a martingale difference sequence relative to the information set $\Im_{t-1}$.

In other words, although $\{X_t(\upsilon)\}$ (the original process in levels) may be uncorrelated over time, its squared version $\{X_t^2(\upsilon)\}$ follows a weak periodic AR(1)-type recursion (in each season υ) with time-varying noise $\eta_t(\upsilon)$. This highlights the potential autocorrelation structure in the squared process that is central to volatility clustering in financial and econometric applications.

Remark 1 Note that $\{X_t\}$ itself is typically uncorrelated over time, but $\{X_t^2\}$ can exhibit significant autocorrelation due to the conditional heteroscedasticity mechanism. Thus, focusing on the squared series is essential for understanding volatility dynamics.

To proceed, define $s\times 1$ vectors $\boldsymbol{X}^2(t) = \left(X_t^2(1), \ldots, X_t^2(s)\right)^\top$, $\boldsymbol{\eta}(t) = (\eta_t(1), dots, \eta_t(s))^\top$, and $\boldsymbol{\omega} = (\omega(1), \ldots, \omega(s))^\top$. Then, Equation (2.3) may be written as

$$\Phi_0 \boldsymbol{X}^2(t) = \boldsymbol{\omega} + \Phi_1 \boldsymbol{X}^2(t-1) + \mathbf{1}\boldsymbol{\eta}(t), \tag{2.4}$$

where Φ_0 and Φ_1 are $s\times s$ matrices determined by the periodic coefficients $\alpha(\upsilon)$. In particular, $\Phi_0(i,j) = \delta_{i=j} - \alpha(i)1_{\{j<i\}}$ and $\Phi_1(i,j) = \alpha(i)1_{\{s+i-j=1\}}$, where 1_B is the indicator function in the set B. Equivalently, define $\mathcal{A}(L) = \Phi_0 - \Phi_1 L$, where L is the backshift operator, $L^k \boldsymbol{X}^2(t) = \boldsymbol{X}^2(t-k)$. The structure of Φ_0 (often lower triangular) implies that Equation (2.4) can be viewed as a system of recursive equations or, more briefly, a weak VAR(1) with periodic constraints (Gladyshev, 1961; Tiao and Grupe, 1980). For $s = 4$, we obtain

$$\Phi_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha(2) & 1 & 0 & 0 \\ 0 & -\alpha(3) & 1 & 0 \\ 0 & 0 & -\alpha(4) & 1 \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} 0 & 0 & 0 & \alpha(1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which can be interpreted as a vector AR(1) process, but specialized to capture seasonal shifts each quarter. In many applications, one may focus only on the relevant season-specific coefficients, as the zero blocks highlight the recursive nature of the model.

THEOREM 2.1 A unique causal, strictly stationary, and ergodic solution to the system stated in Equation (2.4) exists if and only if $\det(\Phi_0 - \Phi_1 z) \neq 0 \quad$ for all complex $|z| \leq 1$, or equivalently $\rho(\Phi) < 1$, where $\Phi = \Phi_0^{-1}\Phi_1$. Under this condition, the solution admits the convergent MA(∞) expansion stated as

$$\boldsymbol{X}^2(t) = \Phi_0^{-1}\boldsymbol{\omega} + \sum_{k=0}^{\infty} \Phi^k \Phi_0^{-1} \boldsymbol{\eta}(t-k). \tag{2.5}$$

Proof See details in Appendix. ■

Remark 2 Gladyshev (1961), Aknouche and Bentarzi (2008), and Lee and Shin (2010) showed that, if the product of the seasonal coefficients $\alpha(\upsilon)$ is small, higher-order moments of $\{X_t^2\}$ exist. For instance, it can be proven that $\mathrm{E}[\|\boldsymbol{X}_t^2\|^m] < +\infty$ if and only if $\prod_{\upsilon=1}^{s} \alpha(\upsilon)^m < 1$. This underpins the asymptotic arguments for the estimators we discuss later.

When a strictly stationary and PC solution to the P-ARCH(1) model exists, the process $\{X_t\}$ is uncorrelated across time yet features time-varying conditional variance. In this sense, $\{X_t\}$ may be viewed as a weak white noise with respect to its raw values, but its squared process $\{X_t^2\}$ exhibits strong autocorrelation and seasonality. This section analyzes the second-order properties of $\{X_t^2\}$ by examining the lag-h covariances of the squared series in each season.

2.4 *Seasonal means and autocovariances of the squared process*

Recall that Equation (2.1) describes the dependence of the conditional variance h_t^2 on past values of X_t. However, for the purpose of studying second-order properties, it is helpful to analyze the squared process X_t^2 directly. In this context, we can rewrite the P-ARCH(1) model in terms of squared values as

$$X_t^2(\upsilon) = \omega(\upsilon) + \alpha(\upsilon)X_{t-1}^2(\upsilon) + \eta_t(\upsilon), \quad \upsilon \in \{1, \ldots, s\}, \tag{2.6}$$

where $\eta_t(\upsilon) = X_t^2(\upsilon) - \mathrm{E}[X_t^2(\upsilon)|\Im_{t-1}]$ is a martingale difference sequence satisfying $\mathrm{E}[\eta_t(\upsilon)] = 0$. This formulation in Equation (2.6) highlights the structure of $X_t^2(\upsilon)$ as a periodic autoregressive process, which is essential for deriving second-order statistics and moment-based estimation techniques. Define $\mu_\upsilon = \mathrm{E}\left[X_t^2(\upsilon)\right]$. Taking expectations in Equation (2.6) yields the expression stated as

$$\mu_\upsilon = \omega(\upsilon) + \alpha(\upsilon)\mu_{\upsilon-1}. \tag{2.7}$$

By iterating Equation (2.7) over one full period s and imposing $\mu_0 = \mu_s$, we have $\mu_s = (1 - \prod_{\upsilon=1}^{s} \alpha(\upsilon))^{-1} \sum_{j=1}^{s} (\prod_{\ell=j+1}^{s} \alpha(\ell))\omega(j)$. Then, for $1 \leq \upsilon \leq s-1$, $\mu_\upsilon = (\prod_{j=1}^{\upsilon} \alpha(j))\mu_s + \sum_{j=1}^{\upsilon} (\prod_{l=j+1}^{\upsilon} \alpha(l))\omega(j)$. Hence, each season υ in the squared process has its own mean μ_υ, reflecting the periodic intercept and slope parameters.

To derive the autocovariances of the squared series, consider the centered process $Y_t^2(\upsilon) = X_t^2(\upsilon) - \mu_\upsilon$. From Equation (2.6), it follows that

$$Y_t^2(\upsilon) = \alpha(\upsilon)Y_t^2(\upsilon - 1) + \eta_t(\upsilon), \quad \upsilon \in \{1, \ldots, s\}. \tag{2.8}$$

Multiplying both sides of Equation (2.8) by $X_t^2(\upsilon - h)$ (for $h \geq 0$) and taking expectations yields a recursion for the seasonal autocovariances given by

$$\gamma_\upsilon(h) = \mathrm{Cov}[X_t^2(\upsilon), X_t^2(\upsilon - h)] = \begin{cases} \sigma_\eta^2(\upsilon), & h = 0; \\ \alpha(\upsilon)\gamma_{\upsilon-1}(h-1), & h \geq 1; \end{cases}$$

where $\sigma_\eta^2(\upsilon) = \mathrm{E}[\eta_t^2(\upsilon)]$. By definition, $\eta_t(\upsilon) = h_t^2(\upsilon)(e_t^2(\upsilon) - 1)$, and with $\mathrm{E}[e_t^2(\upsilon)] = 1$, we obtain $\sigma_\eta^2(\upsilon) = \mathrm{E}[\eta_t^2(\upsilon)] = (\kappa_2 - 1)\lambda_\upsilon$, with $\lambda_\upsilon = \mathrm{E}[h_t^4(\upsilon)], \kappa_2 = \mathrm{E}[e_t^4(\upsilon)]$. Hence, for $h \geq 1$, we get $\gamma_\upsilon(h) = (\kappa_2\lambda_\upsilon - \mu_\upsilon^2)1_{\{h=0\}} + (\prod_{j=0}^{h-1} \alpha(\upsilon - j))\sigma_\eta^2(\upsilon - h)$. Note that periodic indices are understood in module s.

2.5 *Fourth-order terms and properties of means and covariances*

To evaluate terms like $\lambda_\upsilon = \mathrm{E}[h_t^4(\upsilon)]$, it is possible to recursively solve $\lambda_\upsilon = \omega^2(\upsilon) + 2\alpha(\upsilon)\omega(\upsilon)\mu_{\upsilon-1} + \kappa_2\alpha^2(\upsilon)\lambda_{\upsilon-1}$. Imposing $\lambda_0 = \lambda_s$, it yields the expression given by

$$\lambda_s = \left(1 - \prod_{\upsilon=1}^{s} \kappa_2\alpha^2(\upsilon)\right)^{-1} \sum_{j=1}^{s} \kappa_2^{s-j} \left(\prod_{\ell=j+1}^{s} \alpha^2(\ell)\right) \left(\omega^2(j) + 2\alpha(j)\omega(j)\mu_{j-1}\right),$$

and for $1 \leq \upsilon \leq s-1$, we have that

$$\lambda_\upsilon = \sum_{j=1}^{\upsilon} \kappa_2^{\upsilon-j} \left(\prod_{l=j+1}^{\upsilon} \alpha^2(l)\right) \left(\omega^2(j) + 2\alpha(j)\omega(j)\mu_{j-1}\right) + \left(\prod_{j=1}^{\upsilon} \kappa_2\alpha^2(j)\right) \lambda_s.$$

Although the process $\{X_t^2(\upsilon)\}$ for each season υ is derived from a P-ARCH(1) structure, its autocovariances need not be symmetric about zero in the usual sense. In fact, the autocovariances satisfy the relation $\gamma_\upsilon(-h) = \gamma_{\upsilon+h}(h)$ for integer $h \geq 0$, due to the periodic indexing. Consequently, each season υ may exhibit a distinct pattern of dependence, meaning that a single stationarity assumption (in the traditional sense) does not hold globally when $s > 1$. Instead, periodic stationarity ensures that the full autocovariance structure is obtained by evaluating $\gamma_\upsilon(h)$ for $\upsilon \in \{1, \ldots, s\}$, and $h \geq 0$. This framework provides a means to analyze how season-specific effects influence persistence and mean reversion in squared returns (or other heteroscedastic time series), reflecting the periodic evolution of conditional variance.

Consider an observed sample $\{X_1, \ldots, X_n\}$ of size $n = sN$ from the unique strictly stationary and PC solution $\{X_t\}$ of the P-ARCH(1) model stated in Equation (2.3). Equivalently, this sample can be interpreted as $\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N\}$ as a realization of a second-order stationary vector process $\{\boldsymbol{X}_t, t \in \mathrm{Z}\}$, where $\boldsymbol{X}_t^2 = (X_t^2(1), \ldots, X_t^2(s))^\top$.

For each season $\upsilon \in \{1, \ldots, s\}$ and integer lag $h \geq 0$, define

$$\widehat{\mu}_\upsilon = \frac{1}{N} \sum_{t=0}^{N-1} X_t^2(\upsilon), \quad \widehat{\gamma}_\upsilon(h) = \frac{1}{N} \sum_{t=0}^{N-1} X_t^2(\upsilon) X_t^2(\upsilon - h) - \widehat{\mu}_\upsilon \widehat{\mu}_{\upsilon-h},$$

where indices are interpreted in module s. Let $\widehat{\boldsymbol{\mu}} = (\widehat{\mu}_1, \ldots, \widehat{\mu}_s)^\top$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_s)^\top$, and $\widehat{\boldsymbol{\gamma}}(h) = (\widehat{\gamma}_1(h), \ldots, \widehat{\gamma}_s(h))^\top, \boldsymbol{\gamma}(h) = (\gamma_1(h), \ldots, \gamma_s(h))^\top$. The following propositions summarize key asymptotic properties of these estimators, under the assumption that the P-ARCH(1) process possesses the necessary finite moments (up to order 2 or 4) and that the vector sequence $\{\boldsymbol{X}_t^2\}$ is second-order stationary and ergodic.

Proposition 2.2 Assume $\{\boldsymbol{X}_t^2\}$ satisfies the P-ARCH(1) representation stated in Equations (2.3)–(2.5) with finite second moments. Then, for each $\upsilon \in \{1, \ldots, s\}$, we have that:

(i) $\widehat{\mu}_\upsilon \xrightarrow{\text{almost sure}} \mu_\upsilon$.
(ii) As $N \to \infty$, $N \operatorname{Cov}[\widehat{\mu}_\upsilon, \widehat{\mu}_{\upsilon'}]$ converges to $(\boldsymbol{V}_{\text{as}})_{\upsilon,\upsilon'}$, where $\boldsymbol{V}_{\text{as}} = \sum_{h \in \mathrm{Z}} \operatorname{Cov}[\boldsymbol{X}_t^2, \boldsymbol{X}_{t-h}^2]$.
(iii) $\lim_{N\to\infty} \mathrm{E}\left[(\widehat{\mu}_\upsilon - \mu_\upsilon)^2\right] = 0$.
(iv) The vector $\sqrt{N}(\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu})$ converges in distribution to a multivariate normal with mean $\boldsymbol{0}$ and covariance matrix $\boldsymbol{V}_{\text{as}}$.

Proof See details in Appendix. ■

Proposition 2.3 Suppose $\{\boldsymbol{X}_t^2\}$ in Equation (2.3)–(2.5) admits moments up to order 4. Then, for any integer $h \geq 0$ and each $\upsilon \in \{1, \ldots, s\}$, we have that:

(i) $\widehat{\gamma}_\upsilon(h) \xrightarrow{\text{almost sure}} \gamma_\upsilon(h)$.
(ii) $\lim_{N\to\infty} N \operatorname{Cov}[\widehat{\gamma}_\upsilon(h), \widehat{\gamma}_{\upsilon'}(k)] = (W_{\text{as}}(h,k))_{\upsilon,\upsilon'}$, where $W_{\text{as}}(h,k) = \sum_{l \in \mathrm{Z}} \operatorname{Cov}[\boldsymbol{X}_t^2 \otimes \boldsymbol{X}_t^2(h), \boldsymbol{X}_{t-l}^2 \otimes \boldsymbol{X}_{t-l}^2(k)]$.
(iii) $\lim_{N\to\infty} \mathrm{E}\left[(\widehat{\gamma}_\upsilon(h) - \gamma_\upsilon(h))^2\right] = 0$.
(iv) The vector $\sqrt{N}(\widehat{\boldsymbol{\gamma}}(h) - \boldsymbol{\gamma}(h))$ converges in distribution to a multivariate normal with mean $\boldsymbol{0}$ and covariance $W_{\text{as}}(h,h)$.

Proof See details in Appendix. ■

These results show that both the empirical means $\{\widehat{\mu}_\upsilon\}$ and the empirical covariances $\{\widehat{\gamma}_\upsilon(h)\}$ are consistent and asymptotically normal under regular conditions on the fourth moments of $\{\boldsymbol{X}_t^2\}$. Consequently, these empirical estimates serve as fundamental building blocks in moment-based estimation methods, including the YW-type approach analyzed later in the article.

3. Estimation via periodic Yule-Walker equations

This section presents the estimation P-ARCH models using the YW approach, which provides a computationally efficient alternative to traditional methods such as LS and QML. Unlike iterative procedures that can be computationally intensive, the YW estimation method leverages empirical means and autocovariances of the squared process, making it particularly suitable for large datasets (Lund and Basawa, 1999). The theoretical foundation of the estimators, their properties, and their implementation are discussed in detail.

3.1 *Definition of the Yule-Walker estimators*

Let $\{\widehat{\mu}_\upsilon\}$ be the empirical seasonal means and $\{\widehat{\gamma}_\upsilon(h)\}$ be the empirical autocovariances of $\{X_t^2(\upsilon)\}$. Recall that we parametrize the P-ARCH(1) model using $\boldsymbol{\theta}(\upsilon) = (\alpha(\upsilon), \omega(\upsilon))^\top$, for $\upsilon \in \{1, \ldots, s\}$. Then, the full parameter vector is given by $\boldsymbol{\theta} = (\boldsymbol{\theta}^\top(1), \ldots, \boldsymbol{\theta}^\top(s))^\top$.

Based on the moment relations in Equation (2.3) and the implied autocovariance structure of the squared series, a natural set of YW-type estimators can be constructed. Specifically, for each season $\upsilon \in \{1, \ldots, s\}$, define

$$\widehat{\alpha}(\upsilon) = \frac{\widehat{\gamma}_\upsilon(2)}{\widehat{\gamma}_{\upsilon-1}(1)}, \quad \widehat{\omega}(\upsilon) = \widehat{\mu}_\upsilon - \widehat{\alpha}(\upsilon)\widehat{\mu}_{\upsilon-1}, \tag{3.9}$$

where indices are interpreted in module s; $\widehat{\gamma}_\upsilon(h)$ is the sample autocovariance of the squared process at lag h for season υ, and $\widehat{\mu}_\upsilon$ is the sample mean. Note that we require $\widehat{\gamma}_{\upsilon-1}(1) \neq 0$ to avoid degeneracy in the ratio for $\widehat{\alpha}(\upsilon)$. These estimators provide closed-form expressions for the intercept $\omega(\upsilon)$ and the ARCH coefficient $\alpha(\upsilon)$ in each season.

Remark 1 Although the YW estimators are straightforward to compute and serve as initial guesses, they may be less efficient than QML or LS estimators, which typically exploit the data via iterative procedures. Nonetheless, the YW approach offers a practical alternative for large datasets or as a starting point in an iterative estimation algorithm.

3.2 *Consistency and asymptotic normality*

The following lemma establishes the strong consistency and asymptotic distribution of the season-specific YW estimates $\widehat{\alpha}(\upsilon)$ and $\widehat{\omega}(\upsilon)$. Let $\widehat{\boldsymbol{\theta}}(\upsilon)$ denote the vector $(\widehat{\alpha}(\upsilon), \widehat{\omega}(\upsilon))^\top$.

Lemma 3.1 Consider the squared P-ARCH(1) process and let $\{\boldsymbol{X}_t^2\}$ satisfy the vector representation in Equation (2.5). Assume the process admits finite moments up to order 4. Then, for each $\upsilon \in \{1, \ldots, s\}$, we have that:

(i) $\widehat{\boldsymbol{\theta}}(\upsilon)$ converges almost surely to $\boldsymbol{\theta}(\upsilon)$.

(ii) $\sqrt{N}(\widehat{\boldsymbol{\theta}}(\upsilon) - \boldsymbol{\theta}(\upsilon))$ converges in distribution to a normally distributed vector $\mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\text{as}}(\upsilon))$, where $\boldsymbol{\Sigma}_{\text{as}}(\upsilon) = \boldsymbol{D}(\upsilon)\widetilde{\boldsymbol{\Sigma}}_{\text{as}}(\upsilon)\boldsymbol{D}(\upsilon)^\top$,

$$\boldsymbol{D}(\upsilon) = \begin{pmatrix} \dfrac{1}{\gamma_{\upsilon-1}(1)} & 0 \\ -\dfrac{\mu_{\upsilon-1}}{\gamma_{\upsilon-1}(1)} & 1 \end{pmatrix},$$

$\widetilde{\boldsymbol{\Sigma}}_{\text{as}}(\upsilon) = \mathrm{E}\left[\eta_t^2(\upsilon)\boldsymbol{Z}_t(\upsilon)\boldsymbol{Z}_t(\upsilon)^\top\right]$, and $\boldsymbol{Z}_t(\upsilon) = \left(X_t^2(\upsilon-2) - \mu_{\upsilon-2}, 1\right)^\top$. Moreover, for $\upsilon \neq \upsilon'$, the sequence $\sqrt{N}(\widehat{\boldsymbol{\theta}}(\upsilon) - \boldsymbol{\theta}(\upsilon))$ is asymptotically independent of $\sqrt{N}(\widehat{\boldsymbol{\theta}}(\upsilon') - \boldsymbol{\theta}(\upsilon'))$.

Proof See details in Appendix. ∎

Since each season is handled separately, it follows that the full parameter vector $\boldsymbol{\theta}$ can be estimated by stacking all season-specific estimators. The next theorem collects these results to deliver a joint asymptotic distribution.

THEOREM 3.2 Suppose $\{\boldsymbol{X}_t^2\}$ admits finite moments up to order 4. Let $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\theta}}^\top(1), \ldots, \widehat{\boldsymbol{\theta}}^\top(s))^\top$. Then $\sqrt{N}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{\mathcal{D}} \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\text{as}}(\boldsymbol{\theta}))$, where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution to, $\boldsymbol{\Sigma}_{\text{as}}(\boldsymbol{\theta}) = \boldsymbol{D}(\boldsymbol{\theta})\, \widetilde{\boldsymbol{\Sigma}}_{\text{as}}(\boldsymbol{\theta})\, \boldsymbol{D}(\boldsymbol{\theta})^\top$, and both $\boldsymbol{D}(\boldsymbol{\theta})$ and $\widetilde{\boldsymbol{\Sigma}}_{\text{as}}(\boldsymbol{\theta})$ are $2s \times 2s$ block matrices whose v-th diagonal block corresponds to $\mathbf{D}(v)\boldsymbol{\Sigma}_{\text{as}}(v)\mathbf{D}^\top(v)$ from Lemma 3.1.

Proof See details in Appendix. ■

In practice, the YW estimators stated in Equation (3.9) can be applied directly or used as initial values in more complex estimation algorithms such as QML or LS. Additionally, bootstrap techniques can be incorporated to refine inference about the distribution of these estimators, often by resampling the residuals of the P-ARCH(1) model. However, for purely theoretical analysis or initial exploration, the YW approach stands out due to its analytic simplicity and the explicit form of its asymptotic variance.

It is important to note that, unlike in P-AR processes where YW estimators can achieve optimal asymptotic efficiency under Gaussian assumptions (Pagano, 1978), the heteroscedastic nature of P-ARCH(1) weakens that property. Nevertheless, the YW method remains a straightforward and computationally inexpensive option in the periodic ARCH setting.

3.3 *Wald test for periodic parameters*

To apply Theorem 3.2, consider testing a linear hypothesis on the parameter vector $\boldsymbol{\theta}$. Let

$$\text{H}_0\colon R\,\boldsymbol{\theta} = \boldsymbol{\theta}_0 \quad \text{versus} \quad \text{H}_1\colon R\,\boldsymbol{\theta} \neq \boldsymbol{\theta}_0, \tag{3.10}$$

where R is a known $r \times 2s$ matrix of full row rank $r \leq 2s$, and $\boldsymbol{\theta}_0$ is a given $r \times 1$ vector.

The corresponding Wald-type statistic is given by

$$W_N(\widehat{\boldsymbol{\theta}}) = N(R\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)^\top (R\boldsymbol{\Sigma}_{\text{as}}(\boldsymbol{\theta})R^\top)^{-1}(R\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0), \tag{3.11}$$

where $\boldsymbol{\Sigma}_{\text{as}}(\boldsymbol{\theta})$ is the asymptotic covariance matrix from Theorem 3.2 (assumed nonsingular).

Under H_0 in Equation (3.10) and the regularity conditions of Theorem 3.2, $\sqrt{N}(R\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{\mathcal{D}} \mathcal{N}(\mathbf{0}, R\boldsymbol{\Sigma}_{\text{as}}(\boldsymbol{\theta})R^\top)$, implying $(R\boldsymbol{\Sigma}_{\text{as}}(\boldsymbol{\theta})R^\top)$ is nonsingular. Consequently, by the Slutsky theorem, $W_N(\widehat{\boldsymbol{\theta}}) \xrightarrow{\mathcal{D}} \chi^2_r$, and under H_1, the statistic diverges with N, ensuring consistency, where χ^2_r is the chi-square distribution with r degrees of freedom.

To show that using $\widehat{\boldsymbol{\theta}}$ or $\boldsymbol{\theta}$ in Equation (3.11) yields the same limiting distribution, a one-term Taylor expansion is applied as

$$W_N(\widehat{\boldsymbol{\theta}}) - W_N(\boldsymbol{\theta}) = \sqrt{N}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})^\top \frac{1}{\sqrt{N}} \frac{\partial}{\partial \boldsymbol{\theta}} W_N(\boldsymbol{\theta}_*),$$

where $\boldsymbol{\theta}_*$ lies between $\widehat{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}$. Since $\sqrt{N}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{\mathcal{D}} \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\text{as}})$ and the derivative term goes to zero in probability under H_0, both statistics share the same limit law.

For a given significance level $\alpha \in (0,1)$, the critical value $\chi^2_{r,\alpha}$ is obtained from the χ^2_r distribution with upper-tail probability α. Thus, H_0 is rejected if $W_N(\widehat{\boldsymbol{\theta}}) > \chi^2_{r,\alpha}$, and is accepted otherwise. In the scalar case where $r = 1$, this reduces to a t-type statistic, allowing for one- or two-sided tests.

Example 3.3 (Testing periodicity) An important application involves testing for the presence or absence of periodicity in a given P-ARCH(1) model. For instance, if it is hypothesized that certain $\alpha(\upsilon)$ are equal across seasons (or equal to a particular constant α_0), a simple linear hypothesis can be set up as H_0: $\alpha(\upsilon) = \alpha_0, \upsilon \in \{1, \ldots, s\}$, versus$\mathrm{H}_1$: not all $\alpha(\upsilon)$ equal to α_0. To test this hypothesis, the matrix R and the vector $\boldsymbol{\theta}_0$ are constructed to reflect these equalities. If, for each υ, the data strongly suggest that $\alpha(\upsilon) \neq \alpha_0$, the Wald statistic will tend to exceed its critical value, leading to rejection of the null hypothesis and indicating a genuine periodic effect in the model parameters.

4. Numerical illustrations

This section presents a numerical study of the finite-sample properties of the proposed YW estimators for P-ARCH(1) models, comparing them to LS estimates and illustrating a bootstrap-based inference procedure.

4.1 *Monte Carlo simulation and comparison with least squares*

We begin by simulating a P-ARCH(1) process with period $s = 2$, following the structural setting of Equation (2.2). Specifically, let $\{e_t\}$ be an IID Gaussian sequence with mean zero and variance $\sigma^2 = 1$. Then $\kappa_2 = \mathrm{E}[e_t^4] = 3$ and $\kappa_4 = \mathrm{E}[e_t^8] = 105$. We assume $\{\boldsymbol{X}_t^2\}$ has finite second-order moments, which suffices for deriving the asymptotic distribution of the YW estimators. To that end, each season $\upsilon \in \{1, 2\}$ is associated with $\{\omega(\upsilon), \alpha(\upsilon)\}$.

Under periodic stationarity, one can compute:

- μ_υ, the seasonal mean of $X_t^2(\upsilon)$;
- $\sigma^2(\upsilon) = \mathrm{Var}[X_t^2(\upsilon)]$;
- $\gamma_\upsilon(1)$ and $\gamma_\upsilon(2)$, the seasonal autocovariances at lags 1 and 2.

For instance, we have

$$\mu_\upsilon = \frac{\omega(\upsilon) + \alpha(\upsilon)\omega(\upsilon - 1)}{1 - \alpha(1)\alpha(2)}$$

and

$$\sigma^2(\upsilon) = 3\frac{\beta_\upsilon + 3\alpha(\upsilon)\beta_{\upsilon-1}}{1 - 9\alpha^2(1)\alpha^2(2)} - \mu_\upsilon^2,$$

where $\beta_\upsilon = \omega^2(\upsilon) + 2\alpha(\upsilon)\omega(\upsilon - 1)\mu_{\upsilon-1}$. By defining $\gamma_\upsilon(1) = \alpha(\upsilon)\sigma^2(\upsilon - 1)$ and $\gamma_\upsilon(2) = \alpha(\upsilon)\alpha(\upsilon - 1)\sigma^2(\upsilon)$, one obtains closed-form expressions for the theoretical autocovariances of each season.

We generate $N \in \{1000, 2000, 5000\}$ observations for each of 1000 independent replications and estimate the unknown parameter vector $\boldsymbol{\theta}$ using two methods: the YW approach, as defined in Equation (3.9), and LS estimation. To assess the accuracy of these estimators, we substitute the estimates into the asymptotic covariance formulas, such as $\boldsymbol{\Sigma}_{\mathrm{as}}(\upsilon)$, to obtain $\widehat{\boldsymbol{\Sigma}}_{\mathrm{as}}(\upsilon)$, given by

$$\boldsymbol{\Sigma}_{\mathrm{as}}(\upsilon) = \frac{2}{3}\left(1 + \frac{\mu^2(\upsilon)}{\sigma^2(\upsilon)}\right)\begin{pmatrix} \dfrac{1}{\alpha^2(\upsilon - 1)} & -\dfrac{\mu_{\upsilon-1}}{\alpha^2(\upsilon - 1)} \\ -\dfrac{\mu_{\upsilon-1}}{\alpha^2(\upsilon - 1)} & \dfrac{\alpha^2(\upsilon - 1)\sigma^2(\upsilon) + \mu_{\upsilon-1}^2}{\alpha^2(\upsilon - 1)} \end{pmatrix}.$$

The accuracy of each estimator is measured using the root mean square error (RMSE). Let $(\boldsymbol{\theta}(\upsilon))_i$ denote the i-th component of $\boldsymbol{\theta}(\upsilon)$. The RMSE of the YW estimator is given by

$$\mathrm{RMSE}^{(YW)} = \sqrt{\frac{1}{1000}\sum_{m=1}^{1000}\Big((\boldsymbol{\theta}(\upsilon))_i - (\widehat{\boldsymbol{\theta}}^{(YW)}(\upsilon))_i^{(m)}\Big)^2} \quad \text{versus} \quad \mathrm{RMSE}^{*(YW)},$$

where $\mathrm{RMSE}^{*(YW)}$ is the mean of $(\mathrm{Var}_{\mathrm{as}}[(\widehat{\boldsymbol{\theta}}^{(YW)}(\upsilon))_i])^{1/2}$ computed from the empirical asymptotic variance estimates over 1000 replications. In short, this assesses how well the asymptotic variance formulas predict the actual finite-sample variability.

As an illustration of the Wald test, consider a hypothesis of the form $\mathrm{H}_0^{(i)}(\upsilon, \upsilon')$: $(\boldsymbol{\theta}(\upsilon))_i = (\boldsymbol{\theta}(\upsilon'))_i$, for $1 \leq \upsilon, \upsilon' \leq s$ and $i \in \{1, 2\}$. We form the Wald statistic $N((\widehat{\boldsymbol{\theta}}^{(YW)}(\upsilon))_i - (\widehat{\boldsymbol{\theta}}^{(YW)}(\upsilon'))_i)^2 (R\,\boldsymbol{\Sigma}_{\mathrm{as}}(\widehat{\boldsymbol{\theta}}^{(YW)})R^\top)^{-1}$, and reject $\mathrm{H}_0^{(i)}(\upsilon, \upsilon')$ if it exceeds the 95-th quantile of the χ^2_1-distribution. An analogous procedure applies to the LS estimates.

To summarize the estimation procedure, we present Algorithm 4.1 which details the steps for computing residual-based bootstrap replications in the periodic ARCH(1) model under the YW approach.

Algorithm 4.1: Residual bootstrap for a P-ARCH(1) sample.

Require: Original series $\{X_t(\upsilon)\}$ of length $N = sM$; estimates $(\hat{\omega}(\upsilon), \hat{\alpha}(\upsilon))$ for $\upsilon \in \{1, \ldots, s\}$

Ensure: A bootstrap replicate $\{X_t^*(\upsilon)\}$ of the same size

▷ Step 1: Compute residuals

for $\upsilon = 1$ to s **do**
  for $t = 1$ to N **do**
    $\widetilde{h}_t^2(\upsilon) \leftarrow \hat{\omega}(\upsilon) + \hat{\alpha}(\upsilon)X_{t-1}^2(\upsilon)$
    $\widetilde{e}_t(\upsilon) \leftarrow X_t(\upsilon)/\widetilde{h}_t(\upsilon)$
  end for
  Standardize $\widetilde{e}_t(\upsilon)$ to have mean zero and variance 1, obtaining $\widehat{e}_t(\upsilon)$
end for

▷ Step 2: Resample residuals to form a bootstrap series

for $\upsilon = 1$ to s **do**
  for $t = 1$ to N **do**
    Draw $e_t^*(\upsilon)$ with replacement from $\{\widehat{e}_1(\upsilon), \ldots, \widehat{e}_N(\upsilon)\}$
    $h_t^{*2}(\upsilon) \leftarrow \hat{\omega}(\upsilon) + \hat{\alpha}(\upsilon)(X_{t-1}^*(\upsilon))^2$
    $X_t^*(\upsilon) \leftarrow h_t^*(\upsilon) \cdot e_t^*(\upsilon)$
  end for
end for
return $\{X_t^*(\upsilon)\}, t \in \{1, \ldots, N\}, \upsilon \in \{1, \ldots, s\}$

We compare the bootstrap distributions with asymptotic results, as shown in tables and figures. Columns labeled YW refer to direct asymptotic inference for the YW estimator, while bootstrap-YW corresponds to the residual-based bootstrap estimates. Similarly, LS and bootstrap-LS represent LS estimates and their bootstrap counterparts.

The numerical experiments indicate that the RMSE of the YW estimators generally align closely with the asymptotic standard deviations when $N \geq 2000$, supporting the validity of the theoretical variance formulas. The bootstrap approach provides interval estimates and p-values that match the asymptotic theory well, particularly for larger samples.

While LS estimation often results in slightly lower variance and, consequently, greater efficiency, the YW method remains an attractive alternative due to its straightforward implementation and sufficiently good performance at moderate sample sizes. Overall, the YW estimators balance simplicity and accuracy, and the residual-based bootstrap serves as a valuable tool for improving inference when the sample size is limited.

Table 1 summarizes the simulation results for both YW and LS estimators, along with their respective bootstrap versions. Each cell reports the mean parameter estimate over 1000 replications and the corresponding RMSE values. The columns under bootstrap-YW ($\widehat{\boldsymbol{\theta}}_B^{(YW)}$) and bootstrap-LS' ($\widehat{\boldsymbol{\theta}}_B^{(LS)}$) confirm that the bootstrap-based estimates closely match the direct YW and LS estimates. Also the empirical RMSE values are very close to those predicted by the asymptotic covariance. Notably, in this setting, the LS estimators exhibit smaller RMSEs than their YW counterparts, reflecting slightly higher efficiency.

Table 1. Comparison of YW and LS estimators, including their bootstrap-based versions, with corresponding RMSE values.

N	$\boldsymbol{\theta}$	$\widehat{\boldsymbol{\theta}}^{(\mathrm{YW})}$	$\widehat{\boldsymbol{\theta}}_B^{(\mathrm{YW})}$	$\widehat{\boldsymbol{\theta}}^{(\mathrm{LS})}$	$\widehat{\boldsymbol{\theta}}_B^{(\mathrm{LS})}$	$\mathrm{RMSE}^{(\mathrm{YW})}$	$\mathrm{RMSE}_B^{(\mathrm{YW})}$	$\mathrm{RMSE}^{(\mathrm{LS})}$	$\mathrm{RMSE}_B^{(\mathrm{LS})}$
1000	$\boldsymbol{\theta}=(0.25, 1.00, 0.75, 1.00)'$	0.2280	0.2310	0.2332	0.2358	0.1173	0.1188	0.1088	0.1078
		1.0417	1.0686	1.0272	1.0351	0.2310	0.2385	0.2040	0.2055
		0.6886	0.6999	0.7264	0.7347	0.4013	0.4105	0.2999	0.3033
		1.0923	1.1208	1.0297	1.0350	0.5773	0.6056	0.4076	0.4182
2000		0.2350	0.2380	0.2395	0.2416	0.0990	0.1004	0.0971	0.0970
		1.0286	1.0507	1.0174	1.0262	0.1981	0.2051	0.1859	0.1905
		0.6963	0.7063	0.7250	0.7326	0.3232	0.3309	0.2597	0.2656
		1.0838	1.1073	1.0368	1.0432	0.4635	0.4852	0.3563	0.3730
5000		0.2391	0.2415	0.2443	0.2459	0.0840	0.0859	0.0915	0.0923
		1.0220	1.0367	1.0101	1.0163	0.1702	0.1762	0.1827	0.1863
		0.7120	0.7192	0.7291	0.7334	0.2529	0.2566	0.2098	0.2112
		1.0587	1.0760	1.0315	1.0390	0.3676	0.3804	0.2977	0.3034
1000	$\boldsymbol{\theta}=(0.15, 0.50, 1.00, 1.50)'$	0.1390	0.1398	0.1431	0.1441	0.0789	0.0792	0.0556	0.0545
		0.5228	0.5314	0.5116	0.5134	0.1736	0.1760	0.1134	0.1129
		0.9350	0.9431	0.9715	0.9806	0.5233	0.5323	0.4037	0.4029
		1.5528	1.5743	1.5181	1.5215	0.4211	0.4291	0.3046	0.3063
2000		0.1428	0.1436	0.1463	0.1470	0.0640	0.0642	0.0497	0.0493
		0.5151	0.5215	0.5062	0.5080	0.1421	0.1438	0.1038	0.1049
		0.9442	0.9511	0.9717	0.9785	0.4036	0.4100	0.3344	0.3376
		1.5485	1.5649	1.5226	1.5259	0.3230	0.3278	0.2529	0.2596
5000		0.1445	0.1452	0.1479	0.1484	0.0486	0.0489	0.0400	0.0400
		0.5123	0.5163	0.5038	0.5049	0.1087	0.1097	0.0861	0.0866
		0.9624	0.9673	0.9790	0.9818	0.3079	0.3095	0.2621	0.2601
		1.5325	1.5447	1.5178	1.5215	0.2470	0.2503	0.2046	0.2048
1000	$\boldsymbol{\theta}=(0.50, 1.00, 0.50, 1.00)'$	0.4414	0.4511	0.4564	0.4647	0.2394	0.2469	0.2189	0.2222
		1.1095	1.1632	1.0736	1.0972	0.4353	0.4718	0.3827	0.4015
		0.4481	0.4593	0.4714	0.4812	0.2489	0.2583	0.2284	0.2356
		1.0999	1.1497	1.0481	1.0678	0.4533	0.4940	0.4008	0.4277
2000		0.4562	0.4651	0.4699	0.4769	0.2063	0.2125	0.2033	0.2077
		1.0824	1.1261	1.0518	1.0740	0.3836	0.4139	0.3669	0.3896
		0.4540	0.4639	0.4715	0.4798	0.2071	0.2159	0.2032	0.2102
		1.0916	1.1325	1.0536	1.0724	0.3782	0.4129	0.3626	0.3898
5000		0.4655	0.4733	0.4818	0.4872	0.2027	0.2119	0.2340	0.2402
		1.0668	1.0987	1.0322	1.0479	0.3830	0.4155	0.4405	0.4626
		0.4655	0.4735	0.4759	0.4811	0.1749	0.1808	0.1770	0.1813
		1.0683	1.1001	1.0464	1.0627	0.3276	0.3531	0.3289	0.3452

Figure 1 displays smoothed histograms of the empirical distribution of $\sqrt{N}(\widehat{\theta}_n(i) - \theta(i))$ for the YW estimator (solid line) and its bootstrap version (dashed line) at $N = 1000$. A χ^2 goodness-of-fit test confirms that both histograms are compatible with normality, numerically demonstrating the asymptotic validity of the bootstrap for the YW estimator in this periodic ARCH context. Overall, these results illustrate how the periodic YW approach, paired with a residual-based bootstrap, can yield reliable inference in finite samples for simple P-ARCH(1) processes.

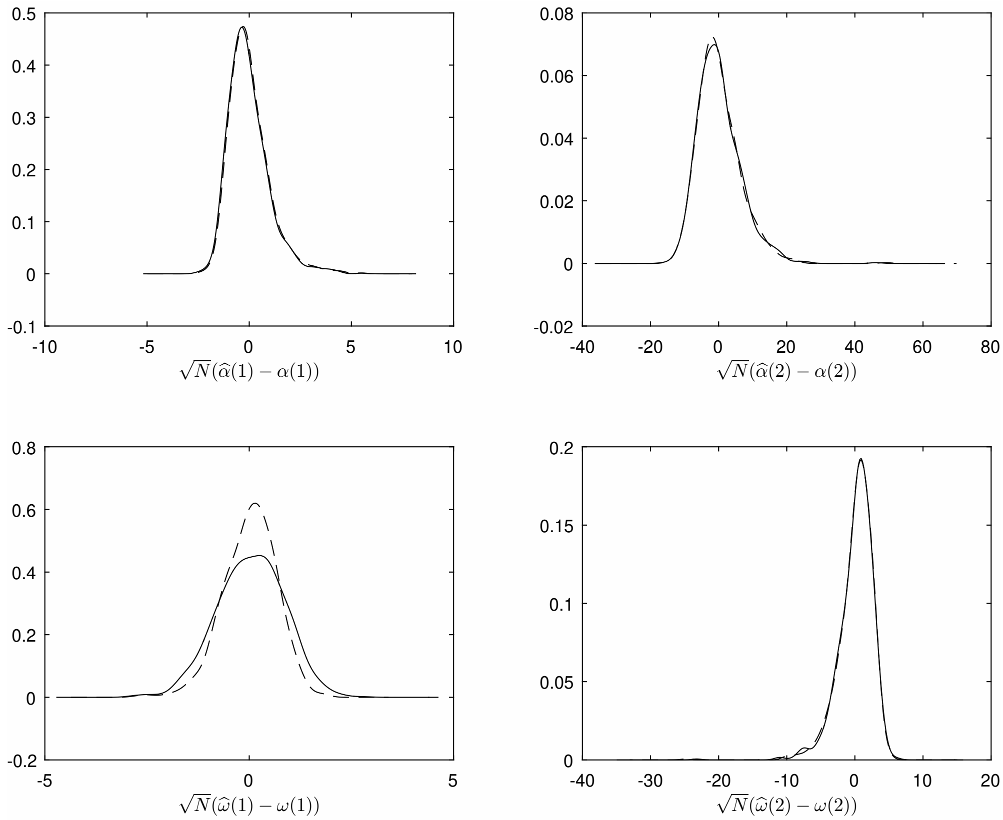

Figure 1. Smoothed histograms of $\sqrt{N}(\widehat{\theta}_n(i) - \theta(i))$ for the YW estimator (solid) and its bootstrap version (dashed) at $N = 1000$.

4.2 *Empirical application*

We illustrate the periodic ARCH methodology with an empirical study of monthly log-returns of Intel Corporation, covering 25 years from January 1973 to December 1997. These data, originally analyzed by Tsay (2002), consist of $12 \times 25 = 300$ observations. For convenience, we label each month by $\upsilon = 1$ (January) through $\upsilon = 12$ (December). Figures 2 and 3 display the time series and monthly boxplots, respectively, revealing apparent differences in volatility across months and moderate outliers.

From Figure 3, it is evident that the distribution of returns varies across months, suggesting the presence of seasonal (or periodic) volatility. Previous analyses by Tsay (2002) showed that the series is weakly uncorrelated in levels but exhibits strong dependence in absolute and squared returns, motivating the use of heteroscedastic models. Our preliminary checks of periodic sample autocorrelations $\widehat{\rho}_\upsilon(h)$ confirmed that a single, non-periodic ARCH(1) process does not adequately capture the apparent month-to-month changes in the correlation structure. Hence, a periodic ARCH(1) model with a distinct intercept for each month is warranted.

Let $r_t(\upsilon)$ be the log-return in month υ of year t. We consider the P-ARCH(1) specification

$$\begin{aligned} r_t(\upsilon) &= m_\upsilon + a_t(\upsilon), \\ a_t(\upsilon) &= h_t(\upsilon)e_t(\upsilon), \quad h_t^2(\upsilon) = \alpha_0(\upsilon) + \alpha_1(\upsilon)a_{t-1}^2(\upsilon), \end{aligned} \tag{4.12}$$

where $\{e_t(\upsilon)\}$ are IID random shocks with zero mean and unit variance. For convenience, we center the data by letting $X_t(\upsilon) = r_t(\upsilon) - \overline{r}(\upsilon)$, where $\overline{r}(\upsilon) = (1/25)\sum_{t=0}^{24} r_t(\upsilon)$.

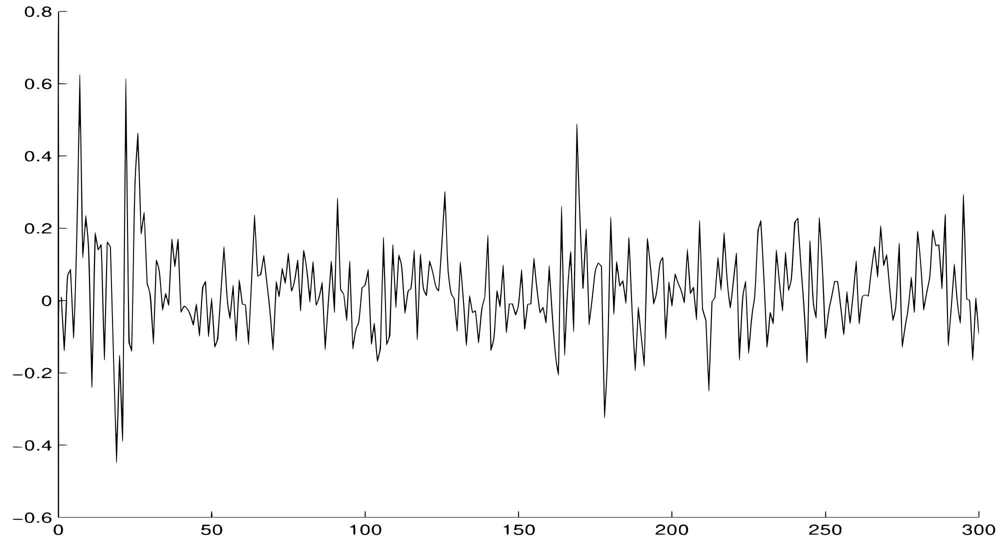

Figure 2. Monthly log-stock returns of Intel Corporation, January 1973–December 1997.

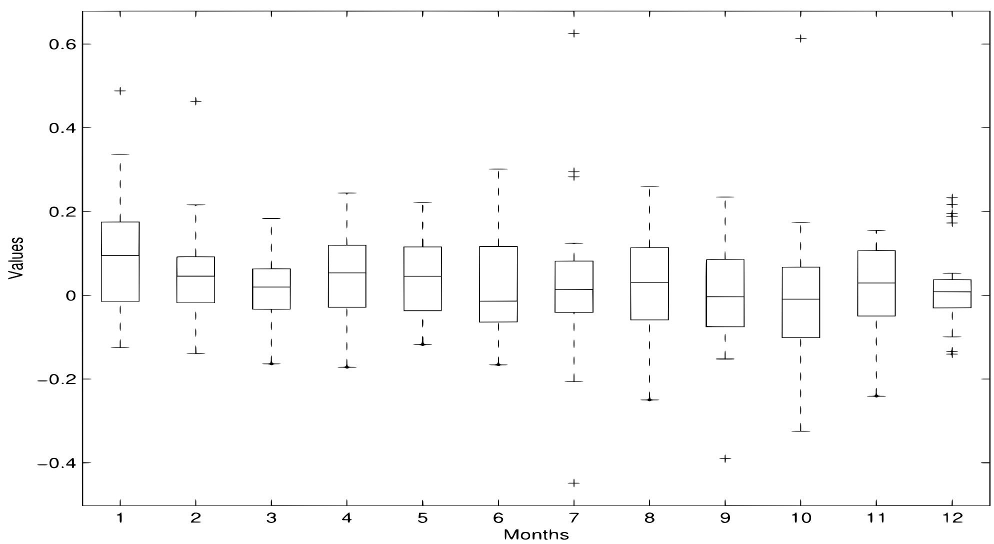

Figure 3. Boxplots of monthly log-returns of Intel Corporation, by month.

Then, we rewrite Equation (4.12) in the form

$$X_t(\upsilon) = \epsilon_t(\upsilon)\sqrt{1+\beta(\upsilon)X_{t-1}^2(\upsilon)}, \quad \text{where} \quad \beta(\upsilon) = \frac{\alpha_1(\upsilon)}{\alpha_0(\upsilon)}, \quad \sigma^2(\upsilon) = \alpha_0(\upsilon). \tag{4.13}$$

Although the parameters in Equations (4.12) and (4.13) differ slightly (due to the intercept and centering), they are straightforwardly related: $\beta(\upsilon) = \alpha_1(\upsilon)/\alpha_0(\upsilon)$ and $\sigma^2(\upsilon) = \alpha_0(\upsilon)$.

Applying the YW procedure to each month yields estimates $\widehat{\beta}(\upsilon)$ and corresponding standard errors (SEs). Table 2 reports the results for the 12 months ($\upsilon \in \{1, \ldots, 12\}$). Observe that some estimated coefficients are quite large (for example, $\widehat{\beta}(9) \approx 4.475$ and $\widehat{\beta}(10) \approx 15.555$), suggesting stronger ARCH effects in certain months. This pattern highlights the importance of modeling the data with distinct month-specific parameters rather than imposing a single global intercept and slope for all months.

Table 2. YW estimates of $\beta(\upsilon)$ for Intel monthly log-returns, with SEs in parentheses.

υ	1	2	3	4	5	6	7	8	9	10	11	12
$\widehat{\beta}(\upsilon)$	0.520	0.006	0.149	0.419	0.499	2.029	1.403	0.458	4.475	15.555	0.101	0.697
SE$[\widehat{\beta}(\upsilon)]$	0.001	0.001	0.003	0.003	0.002	0.008	0.006	0.001	0.011	0.291	0.005	0.002

Although the P-ARCH(1) model stated in Equation (4.12) (or its reparametrization in Equation (4.13)) entails multiple parameters, it yields promising out-of-sample forecasting results for the Intel returns. Figure 4 contrasts the observed log-returns (solid line) with the fitted values (dotted line), demonstrating the model's ability to track the month-to-month volatility changes. Additionally, multi-step-ahead forecasts confirm that the model captures the evolving seasonal volatility pattern.

Table 3 illustrates representative root mean square forecast errors ($RMSE_v(h)$) at various horizons h, showing how the P-ARCH(1) model accommodates the heteroscedastic behavior from month to month. The relatively small forecast errors and their variation across months highlight the importance of modeling season-specific parameters in capturing the cyclical nature of volatility.

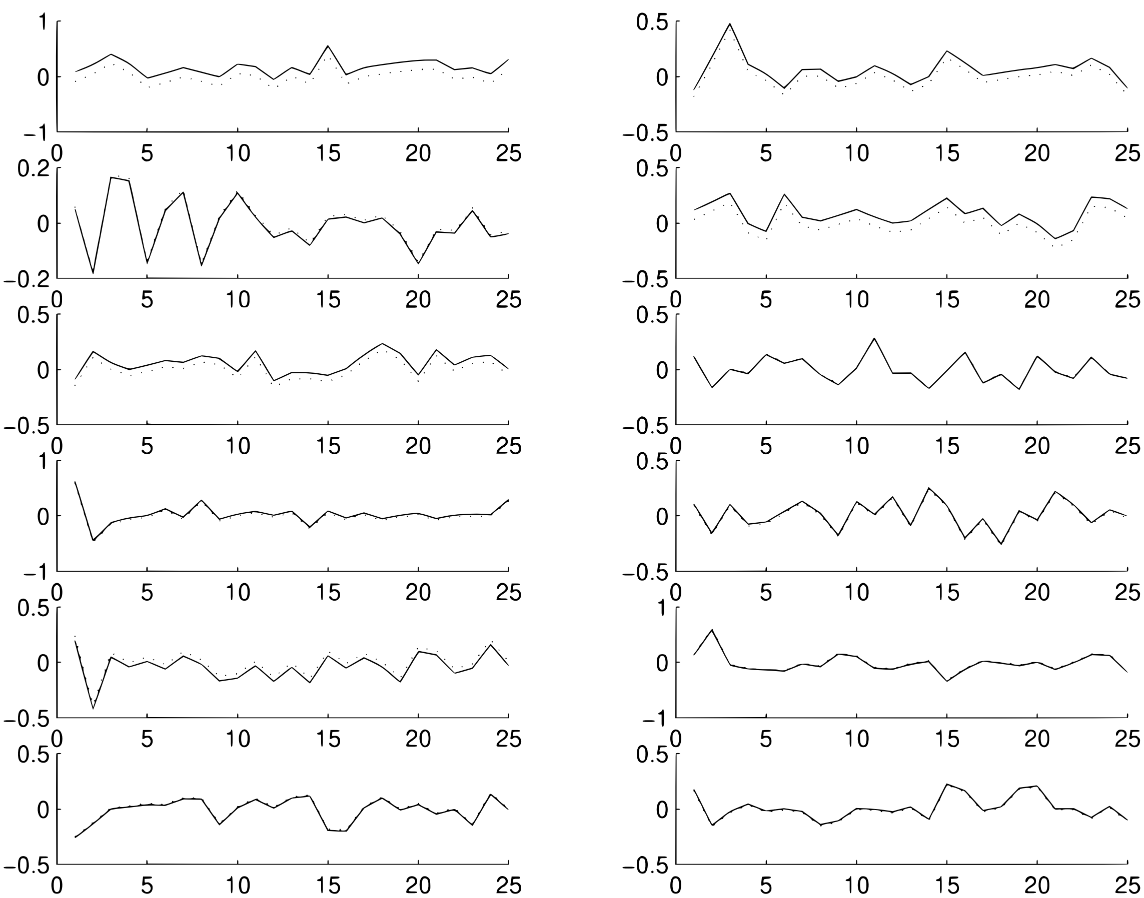

Figure 4. Comparison of observed (solid line) and fitted (dotted line) monthly returns by month for the Intel dataset.

It is noteworthy in Figure 4 that the fitted series tracks the main fluctuations in the data reasonably well, indicating a good model fit. Accordingly, the multi-step forecast errors in Table 3 show no major deterioration across horizons, underscoring the model capability to adapt monthly parameters for evolving volatility patterns.

Table 3. Root mean square forecast errors, $\mathrm{RMSE}_{\upsilon}(h)$, for the h-step-ahead predictions from the fitted P-ARCH(1) model. Each row corresponds to a month $\upsilon \in \{1, \ldots, 12\}$ and columns index the forecasting horizon h.

$\upsilon \backslash h$	1	2	3	4	5	6	7	8	9	10
1	0.1220	0.1274	0.1244	0.1152	0.0616	0.1248	0.1368	0.1313	0.0899	0.1498
2	0.1346	0.1309	0.1218	0.1330	0.1330	0.1132	0.0889	0.0912	0.0563	0.0565
3	0.0845	0.1103	0.1088	0.0824	0.0903	0.0911	0.1330	0.0872	0.1504	0.0940
4	0.2259	0.2170	0.2212	0.2542	0.2284	0.2485	0.1747	0.1471	0.2318	0.2067
5	0.1217	0.1207	0.1258	0.1180	0.1128	0.1378	0.1365	0.1574	0.1284	0.1513
6	0.1279	0.1154	0.2033	0.1380	0.1525	0.1619	0.1740	0.1632	0.1934	0.2229
7	0.1034	0.0993	0.0915	0.0907	0.0879	0.1084	0.1030	0.0943	0.0976	0.0940
8	0.1036	0.0992	0.0978	0.1205	0.1259	0.0835	0.1181	0.0655	0.1010	0.1677
9	0.1382	0.1349	0.1296	0.1322	0.0981	0.1082	0.1327	0.1416	0.2286	0.1259
10	0.1133	0.1157	0.1154	0.1152	0.1171	0.0736	0.1142	0.1463	0.1322	0.2933
11	0.2202	0.2078	0.2064	0.1933	0.1887	0.2021	0.1911	0.2186	0.2046	0.2545
12	0.1445	0.1401	0.2498	0.2275	0.1985	0.2080	0.1021	0.1161	0.1154	0.4007

5. Conclusions

This article has focused on the theoretical and asymptotic properties of first- and second-order moments for a P-ARCH(1) model, as well as the asymptotic behavior of its parameter estimators. Specifically, we established conditions for periodic stationarity, derived moment-based Yule-Walker estimators, and examined their consistency and asymptotic normality. In addition, the theoretical developments allowed us to construct a Wald-type test for evaluating linear restrictions and periodic hypotheses on the model coefficients.

Through Monte Carlo simulations and an empirical illustration, we highlighted the benefits of explicitly modeling season-specific volatility in financial data, emphasizing how strong seasonal or cyclical patterns motivate a periodic framework. The results reinforce the practical utility of P-ARCH(1) models, demonstrating their ability to capture recurring shifts in volatility that may be overlooked by models with time-invariant parameters.

Despite its usefulness, the P-ARCH(1) model (and more generally the P-ARCH framework) has certain limitations as follows:

- In real-world applications, identifying the appropriate period s or higher orders (p, q) can be non-trivial, especially if the seasonal cycle is not clear or if there are multiple overlapping periodicities.
- As s increases or the order p, q becomes larger, the number of parameters can grow, potentially causing estimation challenges (for example, higher variance of estimators, risk of overfitting).
- Our derivations assume that $\{e_t\}$ has finite moments of sufficient order. Financial returns often exhibit heavy tails and potential outliers, so robust or heavy-tailed extensions could be necessary in practice.
- We assumed that the periodic structure and period length are exogenously known or well-defined (for example, monthly patterns). In some datasets, the seasonal or cyclical component may shift or evolve over time, requiring adaptations (such as time-varying periodicity).

Our Yule-Walker approach for P-ARCH(1) models can be extended to address several interesting directions: While we concentrated on the simplest specification with one ARCH term per season, the same moment-based estimation can be adapted to more general P-ARCH(p, q) models or to mixed ARMA–P-ARCH frameworks. These generalizations allow for more complex volatility dynamics and autocorrelation structures in the returns; Many financial and economic time series exhibit long-memory and structural changes that may not be fully captured by purely periodic models. Incorporating long-range dependence, or including break tests within the periodic GARCH-type structure, could enhance model flexibility; Leverage effects are common in equity markets, where negative shocks increase volatility more than positive shocks.

Extending P-ARCH to include such asymmetry (for example, using periodic threshold GARCH or EGARCH) could improve accuracy. Given that financial returns often deviate from normality, exploring robust estimation or heavy-tailed error distributions (such as Student-t) within a periodic framework is a promising direction of investigation. Also, in many applications, it is observed that multiple correlated assets subject to simultaneous seasonal patterns (such as, day-of-week or month-of-year effects). Generalizing the current univariate approach to a vector-valued P-ARCH (or P-CCC-GARCH) model would open a range of possibilities for portfolio risk assessment and co-movements analysis.

Overall, our results underscore the importance of explicitly addressing seasonality or periodicity in volatility modeling. The P-ARCH(1) approach, supplemented by a moment-based Yule-Walker estimation strategy, offers a flexible and computationally accessible way to incorporate cyclical behaviors. Future research extending and refining these models —especially in the presence of structural breaks, asymmetric effects, or long-memory— will further enhance the ability to capture realistic patterns in time-varying volatility.

Additionally, an area for further research is to explore the PARCH models for describing some reliability patterns; see Leiva et al. (2024) for an exhaustive bibliography related to cumulative damage models for reliability analysis.

Appendix: Proofs of technical results

Proof [Theorem 2.1] It is established by induction, relying on the recursive formulation of $\boldsymbol{X}_t^2$ in Equation (2.4). The argument proceeds by analyzing the structure of the squared process at order m and demonstrating that the desired properties hold under the given assumptions. In particular, the recursion follows the form stated as

$$\boldsymbol{X}_t^{2\otimes m} = \Phi^{(m)}\boldsymbol{X}_{t-1}^{2\otimes m} + \sum_{i=0}^{m-1} \Phi_i^{(m)}(\boldsymbol{\eta}_t)\boldsymbol{X}_{t-1}^{2\otimes i}, \quad m \geq 1,$$

where each $\Phi_i^{(m)}(\boldsymbol{\eta}_t)$ is defined recursively by

$$\Phi_i^{(m)}(\boldsymbol{\eta}_t) = \begin{cases} \widetilde{\boldsymbol{\omega}} + \widetilde{\boldsymbol{\eta}}_t, & m = 1, i = 0; \\ \Phi, & m = 1, i = 1; \\ (\widetilde{\boldsymbol{\omega}} + \widetilde{\boldsymbol{\eta}}_t) \otimes \Phi_i^{(m-1)}(\boldsymbol{\eta}_t) + \Phi \otimes \Phi_{i-1}^{(m-1)}(\boldsymbol{\eta}_t), & m > 1; \end{cases}$$

with the convention $\Phi_i^{(m)}(\boldsymbol{\eta}_t) = 0$ whenever $i > m$ or $i = 0$, and $\boldsymbol{X}_t^{2\otimes 0} = 1_{(s)}$. A careful analysis of these expansions shows that all required moments and spectral radius conditions align exactly as stated in Theorem 3.2, completing the induction argument. ■

Proof [Proposition 2.2]

The proof follows from the vector representation $(\widehat{\boldsymbol{\theta}}(\upsilon) - \boldsymbol{\theta}(\upsilon))$, for $1 \leq \upsilon \leq s$ and the results in Lemma 3.1, which establish season-specific strong consistency and asymptotic normality for each υ. By stacking these seasonal estimates into a single vector and applying standard block-diagonal covariance arguments, the desired result is obtained. ■

Proof [Proposition 2.3]

Let $\{\boldsymbol{e}_\upsilon, \upsilon \in \{1, \ldots, s\}\}$ be the canonical basis of R^s. Then, item (i) follows upon the observation that, for each $\upsilon \in \{1, \ldots, s\}$, we have $X_t^2(\upsilon) = \boldsymbol{e}_\upsilon^\top \boldsymbol{X}_t^2$. Since $\{\boldsymbol{X}_t^2, t \in \mathrm{Z}\}$ is a strictly stationary and ergodic process in R^s, the almost sure convergence of the empirical mean follows directly.

For item (ii), we observe that

$$\lim_{N\to+\infty} N\,\mathrm{Cov}[\hat{\mu}_v, \hat{\mu}_{v'}] = \lim_{N\to+\infty} \sum_{|h|<N} \left(1 - \tfrac{|h|}{N}\right) \boldsymbol{e}_v^\top \mathrm{Cov}[\boldsymbol{X}_t^2, \boldsymbol{X}_{t-h}^2]\boldsymbol{e}_{v'}.$$

Item (iii) follows by dominated convergence and Theorem 2.1, whereas the covariance limit follows as stated in the proposition.

For item (iv), we show that $\sqrt{N}\,(\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu})$ converges to a normal distribution with mean zero and covariance $\boldsymbol{V}_{\mathrm{as}}$. Define, for any integer $m > 1$,

$$\boldsymbol{U}_t = \sum_{k=0}^{m} \Phi^k \left(\widetilde{\boldsymbol{\eta}}_{t-k} + \widetilde{\boldsymbol{\omega}}\right), \quad \boldsymbol{W}_t = \sum_{k=m+1}^{+\infty} \Phi^k \left(\widetilde{\boldsymbol{\eta}}_{t-k} + \widetilde{\boldsymbol{\omega}}\right).$$

From the above decomposition, we get $\boldsymbol{X}_t^2 = \boldsymbol{U}_t + \boldsymbol{W}_t$. Let

$$\widehat{\boldsymbol{Q}} = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} \left(\boldsymbol{U}_t - \mathrm{E}[\boldsymbol{U}_t]\right), \quad \widehat{\boldsymbol{V}} = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} \left(\boldsymbol{W}_t - \mathrm{E}[\boldsymbol{W}_t]\right).$$

Then, we have $\sqrt{N}\,(\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu}) = \widehat{\boldsymbol{Q}} + \widehat{\boldsymbol{V}}$. Because $\boldsymbol{U}_t$ is an $(m+1)$-dependent stationary sequence, $\widehat{\boldsymbol{Q}}$ admits a traditional central limit theorem; see Theorem 6.4.2 in Brockwell and Davis (1996). Meanwhile, by the Chebyshev inequality, it can be shown that $\widehat{\boldsymbol{V}}$ converges to 0 in probability uniformly in N as $m \to +\infty$. Consequently, $\widehat{\boldsymbol{Q}}$ and $\sqrt{N}(\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu})$ share the same limiting distribution. Letting $m \to +\infty$, it follows that $\widehat{\boldsymbol{Q}}$ converges in distribution to $\mathcal{N}\left(\boldsymbol{0}, \boldsymbol{V}_{\mathrm{as}}\right)$, where $\boldsymbol{V}_{\mathrm{as}} = \sum_{h\in\mathbb{Z}} \mathrm{Cov}[\boldsymbol{X}_t^2, \boldsymbol{X}_{t-h}^2]$, and $\|\boldsymbol{V}_{\mathrm{as}}\| < +\infty$. Analogous arguments apply for the empirical covariance function $\{\widehat{\gamma}_v(h)\}$, using essentially the same $(m+1)$-dependent approximation and ergodicity. ∎

Proof [Lemma 3.1] Consider the vector $\widehat{\boldsymbol{\gamma}}(h)$, defined for each integer $h \geq 0$, as

$$\widehat{\boldsymbol{\gamma}}(h) = \frac{1}{N} \sum_{t=0}^{N-1} \boldsymbol{X}_t^2 \otimes \boldsymbol{X}_t^2(h) - \widehat{\boldsymbol{\mu}} \otimes \widehat{\boldsymbol{\mu}}(h), \tag{5.14}$$

where $\widehat{\boldsymbol{\mu}}(h) = (1/N) \sum_{t=0}^{N-1} \boldsymbol{X}_t^2(h)$ and $\boldsymbol{X}_t^2(h)$ is the lag-h shift $\left(X_t^2(1-h), \ldots, X_t^2(s-h)\right)^\top$. By ergodicity, the first term in Equation (5.14) converges almost surely to $\mathrm{E}[\boldsymbol{X}_t^2 \otimes \boldsymbol{X}_t^2(h)]$, while the second term converges almost surely to $\boldsymbol{\mu} \otimes \boldsymbol{\mu}(h)$. This verifies parts (i), (ii), and (iii) of Lemma 3.1. Notice that $\boldsymbol{\mu} \otimes \boldsymbol{\mu}(h) = -\boldsymbol{\gamma}(h) + \mathrm{E}\left[\boldsymbol{X}_t^2 \otimes \boldsymbol{X}_t^2(h)\right]$. Hence, the asymptotic distribution of $\sqrt{N}\,(\widehat{\boldsymbol{\gamma}}(h) - \boldsymbol{\gamma}(h))$ coincides with that of

$$\frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} \left(\boldsymbol{X}_t^2 \otimes \boldsymbol{X}_t^2(h) - \mathrm{E}\left[\boldsymbol{X}_t^2 \otimes \boldsymbol{X}_t^2(h)\right]\right). \tag{5.15}$$

One can rewrite $\boldsymbol{X}_t^2$ via the decomposition given above distinguishing between the finite-sum part $\{\boldsymbol{U}_t\}$ and the tail part $\{\boldsymbol{W}_t\}$. A straightforward computation shows that the asymptotic distribution of Equation (5.15) is the same as that of

$$\frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} \left(\boldsymbol{U}_t \otimes \boldsymbol{U}_t(h) - \mathrm{E}\left[\boldsymbol{U}_t \otimes \boldsymbol{U}_t(h)\right]\right),$$

as $m \to +\infty$, where $\boldsymbol{U}_t(h)$ denotes the h-lag shift applied to $\boldsymbol{U}_t$.

To see the limiting behavior, fix any $s \times 1$ vector $\boldsymbol{\lambda}$ and define

$$Y_t(h) = \boldsymbol{\lambda}^\top \left(\boldsymbol{U}_t \otimes \boldsymbol{U}_t(h) - \mathrm{E}[\boldsymbol{U}_t \otimes \boldsymbol{U}_t(h)]\right).$$

Then, $\{Y_t(h)\}$ is a stationary $(m+1)$-dependent process with finite variance. Thus, we get

$$\frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} Y_t(h) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \boldsymbol{\lambda}^\top W(h) \boldsymbol{\lambda}\right),$$

where $W(h) = \sum_{k=-m}^{m} W_h(k)$ and $W_h(k) = \mathrm{Cov}[\boldsymbol{U}_t \otimes \boldsymbol{U}_t(h), \boldsymbol{U}_{t-k} \otimes \boldsymbol{U}_{t-k}(h)]$. As $m \to +\infty$, $W(h)$ converges to

$$W_{\mathrm{as}}(h,h) = \sum_{k \in \mathrm{Z}} \mathrm{Cov}[\boldsymbol{X}_t^2 \otimes \boldsymbol{X}_t^2(h), \boldsymbol{X}_{t-k}^2 \otimes \boldsymbol{X}_{t-k}^2(h)],$$

so establishing a limiting Gaussian distribution for every linear functional of $\sqrt{N}(\widehat{\boldsymbol{\gamma}}(h) - \boldsymbol{\gamma}(h))$. By the Cramér–Wold device, this vector converges in distribution to $\mathcal{N}(\mathbf{0}, W_{\mathrm{as}}(h,h))$. ■

Proof [Theorem 3.2]

The proof begins by noting that $\{\boldsymbol{X}_t^2\}$ is an ergodic and strictly stationary process in the periodic sense. As a result, the strong consistency of the YW estimators follows directly from standard arguments on ergodic averages. Specifically, $\widehat{\mu}_\upsilon \to \mu_\upsilon$ almost surely, and $\widehat{\gamma}_\upsilon \to \gamma_\upsilon$ almost surely, ensuring the consistency of both $\widehat{\alpha}(\upsilon)$ and $\widehat{\omega}(\upsilon)$.

To derive the asymptotic normality, first define $\widetilde{\boldsymbol{\theta}}(\upsilon) = (\widehat{\alpha}(\upsilon), \widetilde{\omega}(\upsilon))^\top$, where $\widetilde{\omega}(\upsilon) = \widehat{\mu}_\upsilon - \alpha(\upsilon)\widehat{\mu}_{\upsilon-1}$. From Equation (2.8), we see that

$$\sqrt{N}\left(\widetilde{\omega}(\upsilon) - \omega(\upsilon)\right) = \sqrt{N}\left(\widehat{\mu}_\upsilon - \mu_\upsilon - \alpha(\upsilon)\left(\widehat{\mu}_{\upsilon-1} - \mu_{\upsilon-1}\right)\right).$$

Meanwhile, using

$$\widehat{\mu}_\upsilon - \mu_\upsilon = \frac{1}{N} \sum_{t=0}^{N-1} \left(X_t^2(\upsilon) - \mu_\upsilon\right) = \alpha(\upsilon)\left(\widehat{\mu}_{\upsilon-1} - \mu_{\upsilon-1}\right) + \frac{1}{N} \sum_{t=0}^{N-1} \eta_t(\upsilon),$$

it follows that

$$\sqrt{N}\left(\widetilde{\omega}(\upsilon) - \omega(\upsilon)\right) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} \eta_t(\upsilon).$$

Similarly,

$$\begin{aligned}\sqrt{N}\left(\widehat{\alpha}(\upsilon) - \alpha(\upsilon)\right) &= \frac{\widehat{\gamma}_{\upsilon-1}^{-1}(1)}{\sqrt{N}} \sum_{t=0}^{N-1} X_t^2(\upsilon-2)\left(\eta_t(\upsilon) - \frac{1}{N}\sum_{t=0}^{N-1} \eta_t(\upsilon)\right) \\ &= \frac{\widehat{\gamma}_{\upsilon-1}^{-1}(1)}{\sqrt{N}} \sum_{t=0}^{N-1} \eta_t(\upsilon)\left(X_t^2(\upsilon-2) - \widehat{\mu}_{\upsilon-2}\right).\end{aligned}$$

Defining $\boldsymbol{Z}_t(\upsilon) = \left(X_t^2(\upsilon-2) - \mu_{\upsilon-2}, 1\right)^\top$ and applying the Cramér–Wold device, we reduce the problem to showing that

$$\frac{1}{\sqrt{N}}\sum_{t=0}^{N-1}\eta_t(\upsilon)\lambda^\top \boldsymbol{Z}_t(\upsilon)\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,\lambda^\top\widetilde{\boldsymbol{\Sigma}}_{\mathrm{as}}(\upsilon)\lambda\right),$$

where $\widetilde{\boldsymbol{\Sigma}}_{\mathrm{as}}(\upsilon)=\mathrm{E}\left[\eta_t^2(\upsilon)\boldsymbol{Z}_t(\upsilon)\boldsymbol{Z}_t(\upsilon)^\top\right]$. Note that $\{\eta_t(\upsilon)\lambda^\top\boldsymbol{Z}_t(\upsilon)\}$ is a martingale difference sequence with constant variance $\lambda^\top\widetilde{\boldsymbol{\Sigma}}_{\mathrm{as}}(\upsilon)\lambda$. A standard martingale central limit theorem (Davidson, 1994) completes this part of the argument.

We observe that $\widehat{\omega}(\upsilon)-\widetilde{\omega}(\upsilon)=-\widehat{\mu}_{\upsilon-1}\left(\widehat{\alpha}(\upsilon)-\alpha(\upsilon)\right)$. Thus, by the Slutsky theorem, multiplying out block terms shows that

$$\sqrt{N}\left(\widehat{\boldsymbol{\theta}}(\upsilon)-\boldsymbol{\theta}(\upsilon)\right)=\widehat{D}(\upsilon)\sqrt{N}\left(\widetilde{\boldsymbol{\theta}}(\upsilon)-\boldsymbol{\theta}(\upsilon)\right)\xrightarrow{\mathcal{D}}\mathcal{N}\left(\mathbf{0},\boldsymbol{D}(\upsilon)\widetilde{\boldsymbol{\Sigma}}_{\mathrm{as}}(\upsilon)\boldsymbol{D}(\upsilon)^\top\right),$$

where

$$\widehat{D}(\upsilon)=\begin{pmatrix}\frac{1}{\widehat{\gamma}_{\upsilon-1}(1)} & \frac{\mu_{\upsilon-2}-\widehat{\mu}_{\upsilon-2}}{\widehat{\gamma}_{\upsilon-1}(1)}\\ -\frac{\widehat{\mu}_{\upsilon-1}}{\widehat{\gamma}_{\upsilon-1}(1)} & 1+\frac{\left(\mu_{\upsilon-2}-\widehat{\mu}_{\upsilon-2}\right)\widehat{\mu}_{\upsilon-1}}{\widehat{\gamma}_{\upsilon-1}(1)}\end{pmatrix}.$$

In the limit as $N\to+\infty$, $\widehat{\boldsymbol{D}}(\upsilon)\xrightarrow{\mathcal{P}}\boldsymbol{D}(\upsilon)$, where $\xrightarrow{\mathcal{P}}$ denotes convergence in probability to. ■

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