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Distributions Theory
Research Article

Some additional facts about the unit-Gompertz distribution

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Abstract

In this article, a two-parameter model called unit-Gompertz is studied over the unit interval. We describe its statistical characteristics such as uncertainty measures of Shannon, Tsallis, Renyi, Mathai-Houbold, Kumar and Verma, order statistics, quantile function, maximum likelihood estimation, factorial and characteristic function, moment generating function, and stress-strength analysis. The effectiveness of the studied model is demonstrated by the use of two datasets from the real life. The model has flexible hazard rate shapes, presents better performance and fits the data better than other usual models.

Keywords: Entropy · Gompertz model · Monte Carlo simulation · Reliability analysis · Stochastic order · Stress-strength analysis.

Mathematics Subject Classification: Primary 60E05 · Secondary 62F10.

1. Introduction

Different unit distributions have been used to represent data for fractions and percentages in a variety of domains, including economics, fitness, risks, life studies, recovery and mortality rates, and measurement sciences. When it comes to modeling and obtaining interpretations based on datasets from the aforementioned domains, there is no doubt that the beta (Johnson, 1949) and Kumaraswamy (Kumaraswamy, 1980) distributions immediately come to mind. It is possible that these traditional models are insufficient, which would seriously impede accurate data analysis. As a result, there are an increasing number of studies in the literature that use unit distributions as their basis. A considerable amount of recent scholarly writing has focused on probability distributions that are included in the unit interval.

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Due to their importance in data modeling across numerous areas, including biology, finance, and environmental research, these bounded distributions have attracted the attention. Bounded distributions are essential because many examples in these domains involve data that naturally falls into the unit range $(0, 1)$. Additionally, better data fitting and more accurate parameter estimations are frequently the outcome of using these distributions in modeling, which closely aligns the model with the real data. Regarding bounded models, check Gemeay et al. (2023), Yildirim et al. (2023), Belili et al. (2023) and Chotikapanich et al. (2007) for further information.

It is evident that bounded distributions have a limited range, and as a result, new models are being proposed and this topic is receiving increased attention. The following are some of the novel models: the generalized exponentiated unit-Gompertz distribution (Sindhu et al., 2023), the unit-Weibull distribution (Mazucheli et al., 2018), the mixture of Log-Bilal distributions (Lone et al., 2023), the unit inverse Gaussian distribution (Ghitany et al., 2019), the unit-Lindley distribution (Mazucheli et al., 2019), the unit Gumbel type-II distribution (Shafiq et al., 2023), unit Burr-III (Modi et al., 2020) distribution, the exponential Topp-Leone distribution (Pourdarvish et al., 2015), log-shifted Gompertz distribution (Jodrá et al., 2020), log-Lindley distribution (Gómez-Déniz et al., 2014), the log-extended exponential-geometric distribution (Al-Zaydi et al., 2023), and the logit-slash distribution (Korkmaz et al., 2020). Given that any other limited interval may be transformed from a random variable (RV) specified on (0,1) using a straightforward linear transformation, it is amazing that these models are defined on the unit interval.

The Gompertz model is a widely used generalized exponential distribution in various real-world applications, particularly in the fields of actuarial science and medicine. It exhibits strong correlations with other well-known models, including the double exponential, Gumbel, Weibull, exponential, and generalized logistics models (Willekens, 2001). An exponentially increasing rate of failure throughout the course of a system life is a fundamental component of the Gompertz model. A number of scholars have currently made contributions to the analysis of the statistical methods and characteristics of this kind of model. For example, refer to references: Read (1983), Makany (1991), Rao et al. (1992), Franses (1994), Chen (1997) and Wu et al. (1999). The statistics literature contains a number of models for modeling lifetime data (see, for example, Ferreira et al. (2023), Cordeiro et al. (2022), Chesneau et al. (2022), Ribeiro-Reis et al. (2022), Cordeiro et al. (2019) and Afify et al. (2017)).

Motivated by the aforementioned literature, our goal is to investigate a novel restricted model including a closed form cumulative distribution function with unit interval. The unit-Gompertz (UG) distribution is the name of the suggested model. First, Mazucheli et al. (2019) suggested this model but did not thoroughly characterize every characteristic of the UG model.

We present the UG model for the following reasons:

(i) It has been observed that the survival times of systems or units are typically greater than zero, but their lifespan cannot be considered infinite. This model can be applied to a variety of issues, including public health and the environment.
(ii) It is effective in modeling bathtub hazards, where failure rates increase, peak, and then decrease as several units are replaced.
(iii) It can accommodate multiple points lying within the range of (0, 1), where multiple units may be dropped or replaced in various applications.
(iv) Two real data applications demonstrate its performance compared to other competing lifetime models. Additionally, we provide several mathematical characteristics of the new distribution that can be utilized by statisticians in their future studies.

The article is organized as following. Section 2 presents the UG model and some of its properties. In Section 3, we provide a random number generator for the UG model and certain measures such as its mean, variance, skewness and kurtosis. In this section, the UG model is also introduced for reliability analysis in a multicomponent stress-length framework. Section 4 is devoted to the moments of the UG distribution and related aspects. In Section 5 are presented some entropy measures for the UG model and is also discussed the residual life function for the UG model. In Section 6, we derive the estimation of parameters for the UG distribution under a classical paradigm, conduct a simulation study, and apply our results to two real data applications to illustrate the performance of the proposed model in comparison with other models stated in the literature. Section 7 establishes our concluding remarks and future extensions based on this work.

2. Novel Gompertz distribution

2.1 Context

By using $Y = \exp(-X)$, if X has a two-parameter Gompertz distribution, then the cumulative distribution function (CDF) of Y is written as

$$F\left(y; b, \eta\right) = 1 - \exp\left(\eta\left(1 - \exp(b\,y)\right)\right), b, \eta, y > 0. \tag{2.1}$$

We obtain the UG distribution (Benkhelifa, 2017) with its probability density function (PDF) and CDF given by

$$f\left(y; b, \eta\right) = b\eta y^{-b-1} \exp\left(\eta\left(1 - y^{-b}\right)\right);\; b, \eta > 0, \tag{2.2}$$

$$F\left(y; b, \eta\right) = \exp\left(\eta\left(1 - y^{-b}\right)\right), \tag{2.3}$$

where $y \in (0,1)$ and $b, \eta > 0$ are scale and structure parameters, respectively. Note that $F(y; b, \eta)$ is differentiable and increases between 0 and 1, being in addition that $\lim_{y\to 0} F(y) = 0$ and $\lim_{y\to 1} F(y) = 1$.

In general, reliability is concerned with assessing a system probability of aging or failure. The survival function (SF) of an RV Y is defined as $R(y) = 1 - F(y) = \mathrm{P}(Y > y)$. It may be specified as probability of a system not failing within a specific given time. The SF of the UG distribution is defined by

$$R(y; b, \eta) = 1 - \exp\left(\eta\left(1 - y^{-b}\right)\right). \tag{2.4}$$

The hazard rate function (HRF) $h(y; b, \eta) = f(y; b, \eta)/(1 - F(y; b, \eta))$ is a helpful tool for a lifetime study. The instant failure rate of Y is the probability of a system failing as it has existed to current moment t and it is given as

$$h(y; b, \eta) = \frac{b\eta y^{-b-1} \exp\left(\eta\left(1 - y^{-b}\right)\right)}{1 - \exp\left(\eta\left(1 - y^{-b}\right)\right)}. \tag{2.5}$$

The odd ratio is defined as

$$\Upsilon_y\left(y; b, \eta\right) = \frac{R_y\left(y\right)}{h_y\left(y\right)} = \frac{\left(1 - \exp\left(\eta\left(1 - y^{-b}\right)\right)\right)^2}{b\eta y^{-b-1} \exp\left(\eta\left(1 - y^{-b}\right)\right)}. \tag{2.6}$$

The cumulative HRF of the UG distribution is obtained as

$$H\left(y\right) = \int_0^y h(t; b, \eta)\mathrm{d}t = -\log\left(1 - \exp\left(\eta\left(1 - y^{-b}\right)\right)\right), \tag{2.7}$$

where $R_y\left(y\right)$ and $h_y\left(y\right)$ is defined in Equations (2.4) and (2.5). Equations (2.1)-(2.7) can be evaluated numerically utilizing computing packages like `Maple`, `MATLAB`, `Mathematica`, `Minitab` and `R-project`. The sketches of Equations (2.2) and (2.5) are given in Figures 1 and 2 for values of chosen parameter. Figure 1 demonstrate how parameters b and η influence the UG PDF and demonstrate flexibility of the PDF forms given in Equation (2.2) where small symmetry, skewness, modality and high tails can be calculated. These figures reflect flexibility of the UG distribution. Figure 2 displays the increasing and bathtub pattern of HRFs.

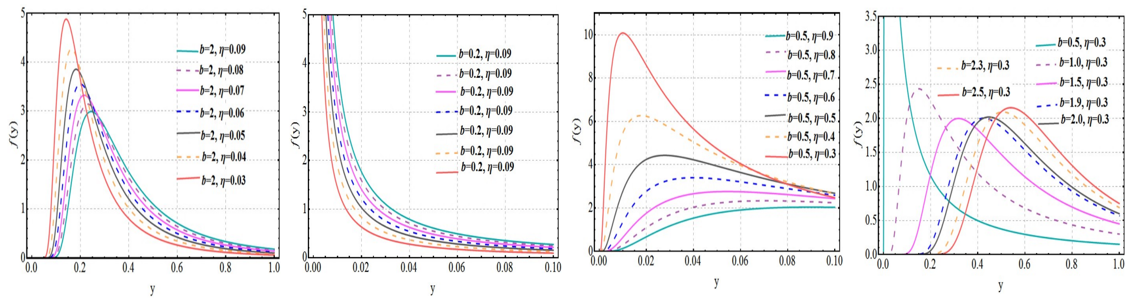


Figure 1.: Plots of the UG PDF at different parameter values.

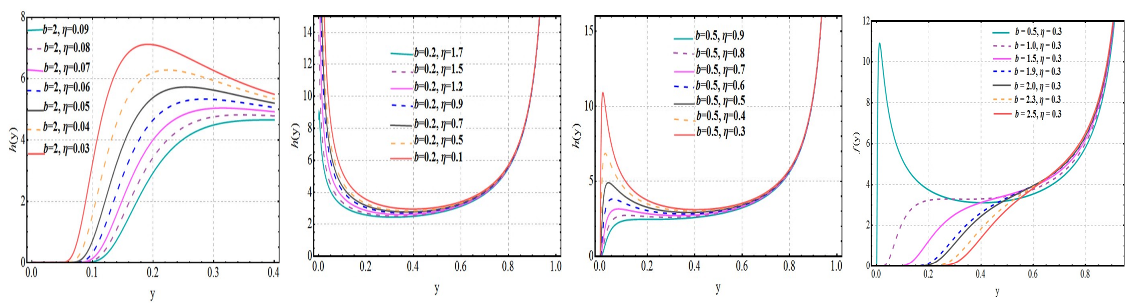


Figure 2.: Plots of the UG HRF at different parameter values.

2.2 Characterization based on the HRF

Suppose $F\left(y; b, \eta\right)$ be an absolute continuous model having PDF $f\left(y; b, \eta\right)$. The HRF referring to $F\left(y; b, \eta\right)$ is defined by

$$h\left(y; b, \eta\right) = \frac{f\left(y; b, \eta\right)}{1 - F\left(y; b, \eta\right)}. \tag{2.8}$$

Here, note that the HRF of a twice differentiable CDF is stated as

$$\frac{h'\left(y; b, \eta\right)}{h\left(y; b, \eta\right)} - h\left(y; b, \eta\right) = k_1\left(y; b, \eta\right), \tag{2.9}$$

where k_1 is a suitable integrable function. For several continuous, univariate distributions given in Equation (2.8) appears to be only differential equation in view of the HRF.

We create a differential equation that has the simplest possible form and not of the trivial form given in Equation (2.9). However, that may not be feasible for certain general distribution families. The characterization of the UG distribution is as follows.

Proposition 1. Let $Y\colon \chi \to (0,1)$ be a continuous RV. Then, the PDF of Y is given in Equation (2.2) iff its HRF $h\,(y;\eta,b)$ satisfies the differential equation formulated as

$$\frac{\mathrm{d}h\,(y;\eta,b)}{\mathrm{d}y} - \Big(\eta b y^{-b-1}\exp\big(-\eta(1-y^{-b})\big)\big(\exp(-\eta(1-y^{-b}))-1\big)^{-1}\Big)h\,(y;\eta,b)$$
$$= -b\eta(b+1)y^{-b-2}\left(\exp(\delta y^{-\eta}\exp(-\upsilon y))-1\right)^{-1}.$$

The proofs of this proposition is in Appendix A.

2.3 Order statistics

Suppose $Y_{(1)} \leq \cdots \leq Y_{(n)}$ be order statistics of a random sample of size n from model. Hence, the PDF of mth order statistics, $Y_{(m)}$ where $m = 1,\ldots,n$, is given by

$$f_{(m)}\,(y;b,\eta) = (y;b,\eta)^{m-1}\,(1-F\,(y;b,\eta))^{n-m}\,f\,(y;b,\eta)\,, \tag{2.10}$$

where $\Psi = n!/\,((m-1)!(n-m)!]$. The PDF is obtained from Equations (2.2), (2.3) and (2.10) and given by

$$f_{(m)}\,(y;b,\eta) = b\eta\Psi\sum_{f=0}^{n-m}\binom{n-m}{f}(-1)^f\,y^{-(b+1)}\exp\Big(\eta\,(f+m-1)\left(1-y^{-b}\right)\Big)\exp\Big(\eta\left(1-y^{-b}\right)\Big),$$

whereas the CDF is $F_{(m)}\,(y;b,\eta) = \sum\limits_{j=m}^{n}\binom{n}{j}F\,(y;b,\eta)^j\,(1-F\,(y;b,\eta))^{n-j}$. Hence, the CDF of the mth order statistic of the UG distribution, $Y_{(m)}$, is expressed as

$$F_{(m)}\,(y;b,\eta) = \sum_{j=m}^{n}\sum_{g=0}^{n-j}(-1)^g\binom{n}{j}\binom{n-j}{g}F\,(y;b,\eta)^{j+g}.$$

In particular, the CDFs of $Y_{(1)}$ and $Y_{(n)}$ are formulated as

$$F_{(1)}\,(y;b,\eta) = 1-\Big(1-\exp\Big(\eta\left(1-y^{-b}\right)\Big)\Big)^n; \tag{2.11}$$

$$F_{(n)}\,(y;b,\eta) = \exp\Big(n\eta\left(1-y^{-b}\right)\Big), \tag{2.12}$$

respectively.

Let $Q_{(m)}$ be (for $0 < \zeta < 1$) the quantile function (QF) of $Y_{(m)}$. Then, from Equations (2.11) and (2.12), we obtain

$$Q_{(1)}\,(y) = Q\left(1-(1-\zeta)^{1/n}\right), \quad Q_{(n)}\,(\zeta) = Q\left(\zeta^{1/n}\right),$$

where the QF of Y is represented by Q. We get an equation for the rth moment of the order statistics denoted as $\rho_r < \infty$. As in Siddiqui and Çağlar (1994), we represent rth moment as

$$\rho_{(m)}^{r} = \mathrm{E}\left(Y_{(m)}^{r}\right) = \sum_{j=n-m+1}^{n} (-1)^{j-n+m-1} \binom{j-1}{n-m} \binom{n}{j} I_j\left(r\right), \tag{2.13}$$

with $I_j\left(r\right) = r\int\limits_0^\infty y^{r-1}\left(1-F\left(y\right)\right)^j \mathrm{d}y$.

Proposition 2. Let $Y_{(1)} \leq \cdots \leq Y_{(n)}$ be the order statistics of a random sample of size n from the UG distribution. The rth moment of the mth order statistic of PDF given in Equation (2.2) can be delineated as

$$\rho_{(m)}^{r} = \mathbb{k}_{j,m,n}\frac{r}{b}\sum_{s=0}^{j}(-1)^s\binom{j}{s}\exp(s\eta)\left(\eta s\right)^{\frac{r}{b}}\Gamma\left(-\frac{r}{b},\eta\right),$$

where $\mathbb{k}_{j,m,n} = \sum_{j=n-m+1}^{n}(-1)^{j-n+m-1}\binom{j-1}{n-m}\binom{n}{j}$. Appendix A contains the proof for this proposition.

2.4 Stochastic ordering

It is of concern to describe, for practical reasons, the underlying Stochastic ordering (St-O) of these members as per parameters if we work with a generalized family of distributions. Here, certain model functions may be utilized as a function of likelihood ratio (LR) function, HRF and CDF. Moreover, we're focusing on likelihood ratios order listed below. For RV Y and X, we say, $Y \succeq_{\mathrm{lr}} X$, if ratio of relevant PDFs are reducing function in x. A significant mechanism for the estimation of relative behavior is the St-O of continuous positive RVs. Let RV $Y > X$ have:

(i) Stochastic order $X \preccurlyeq_{\mathrm{st}} Y$, if $F_X\left(y\right) \preccurlyeq F_Y\left(y\right), \forall y$;
(ii) Hazard rate order $X \preccurlyeq_{\mathrm{hr}} Y$ if $h_X\left(y\right) \succcurlyeq h_Y\left(y\right), \forall y$;
(iii) LR order $X \preccurlyeq_{\mathrm{lr}} Y$, if $f_X\left(y\right)/f_Y\left(y\right)$ reduces in y.

Consider the following implications (Ross et al., 1996):

$$X \preccurlyeq_{\mathrm{lr}} Y \Rightarrow X \preccurlyeq_{\mathrm{hr}} Y \Rightarrow X \preccurlyeq_{\mathrm{st}} Y. \tag{2.14}$$

The UG models are ordered in relation to strongest LR ordering as reported below.

Proposition 3. Let $X \sim \mathrm{UG}\left(\eta_1, b\right)$ and $Y \sim \mathrm{UG}\left(\eta_2, b\right)$. If $b_1 = b_2 = b$ and $\eta_1 \leq \eta_2$, hence $X \leq_{\mathrm{lr}} Y$ $\left(X \leq_{\mathrm{st}} Y, X \leq_{\mathrm{hr}} Y\right)$

Refer to Appendix A for the proof of this proposition.

3. Random number generator, related measures, and reliability

3.1 Random number generator

The function $F\left(y\right)$ is inverted to generate the RV Y as mentioned previously. Let ζ be an RV and $\zeta \sim \mathrm{U}(0,1)$. As a solution of the equation, an observation of Y can then be performed solving $\eta\left(1-y^{-b}\right) = \log\left(\zeta\right)$. Therefore, we have that

$$Y = Q\left(\zeta, b, \eta\right) = \left(1 - \frac{\log\left(\zeta\right)}{\eta}\right)^{-\frac{1}{b}} \sim \mathrm{UG}\left(b, \eta\right). \tag{3.15}$$

Specifically, by placing $\zeta = (0.25, 0.50, 0.75)$ in Equation (3.15), the first, second, and third quartiles are attained. Important quantities of $\mathrm{S}_{B;b,\eta}$ and $\mathrm{K}_{M;b,\eta}$ are provided by $\phi_3 = \mu_3/\sigma^3$ and $\phi_4 = \mu_4/\sigma^4$, respectively, where μ_κ is the foundational κth moment and σ is the standard deviation.

3.2 Related measures

Although for some parametric values, moments of model cannot occur, relevant quantile-based measures $\mathrm{S}_{B;b,\eta}$ and $\mathrm{K}_{M;b,\eta}$ are more appropriate often. For models without moments, such indicators are more reliable and do occur. The indicators related to the Bowley skewness and the Moors kurtosis are defined as

$$\mathrm{S}_{B;b,\eta} = \frac{Q\left(6/8; b, \eta\right) + Q\left(2/8; b, \eta\right) - 2Q\left(4/8; b, \eta\right)}{Q\left(6/8; b, \eta\right) - Q\left(2/8; b, \eta\right)},$$

$$\mathrm{K}_{M;b,\eta} = \frac{Q\left(7/8; b, \eta\right) - Q\left(5/8; b, \eta\right) + Q\left(3/8; b, \eta\right) - Q\left(1/8; b, \eta\right)}{Q\left(6/8; b, \eta\right) - Q\left(2/8; b, \eta\right)}.$$

Therefore, if $\mathrm{S}_{B;b,\eta} < 0$, $\mathrm{S}_{B;b,\eta} > 0$, and $\mathrm{S}_{B;b,\eta} = 0$, then the distribution is called left, right skewed, and symmetrical. Respectively, a higher value of $\mathrm{K}_{M;b,\eta}$ means a strong tail for model and a small finding of $\mathrm{K}_{M;b,\eta}$ implies a mild tail conversely. Figures 3, 4 and 5 display the 3D and contour graphs of mean, variance and median behavior in comparison to UG distribution and the estimates of parameters. In the context of the UG model and according to parametric values, Figures 6 and 7 graphically analyses behavior of $\mathrm{S}_{B;b,\eta}$ and $\mathrm{K}_{M;b,\eta}$.

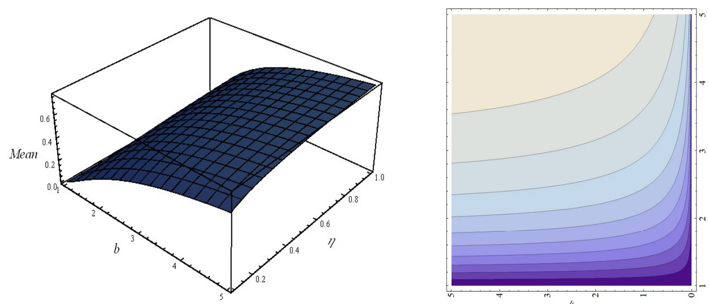


Figure 3.: 3D and contour curves of mean of the UG distribution.

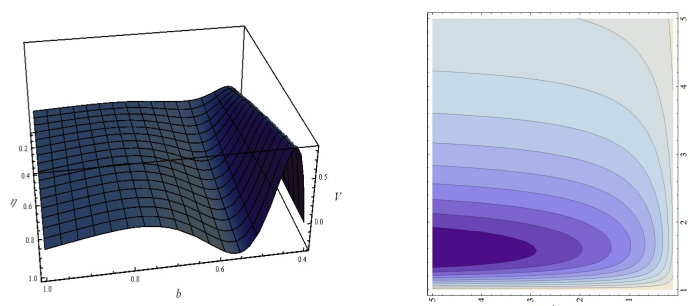


Figure 4.: 3D and contour graphs of variance of the UG distribution.

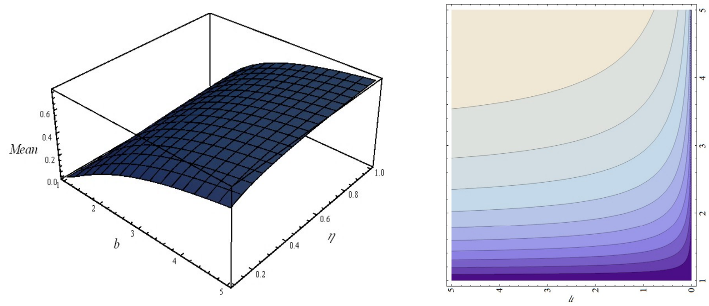


Figure 5.: 3D and contour graphs for median of the UG distribution.

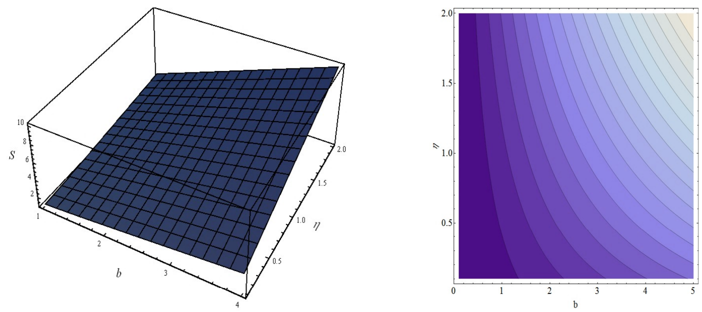


Figure 6.: Graphs of skewness function and contour plot of skewness function.

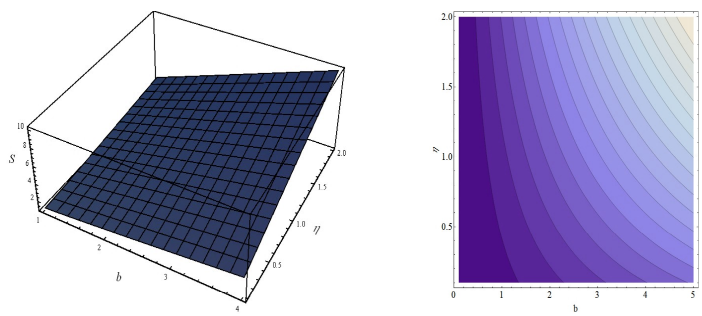


Figure 7.: Graphs of kurtosis function and contour plot of kurtosis function.

3.3 Reliability in a multicomponent stress-strength framework

The stress-strength distribution in concept of reliability provides scheme of survival of a device. Let $Y, Y_1, \ldots, Y_k$ be random samples which makes F_X to be the CDF of common stress, with Y, and $Y_1, \ldots, Y_k$ being independent and identically distributed having CDF, F_Y, subject to X. Thus, reliability in multicomponent stress-strength distribution is written as

$$\begin{aligned} R_{s,k} &= \mathrm{P}\left(\text{at least one } Y_1, \ldots, Y_k \text{ is greater than } X\right) \\ &= \sum_{i=s}^{k} \binom{k}{i} \int\limits_{-\infty}^{\infty} \left(1 - F_Y\left(x; \cdot\right)\right)^i \left(F_Y\left(x; \cdot\right)\right)^{k-i} \mathrm{d}F_X\left(x; \cdot\right). \end{aligned} \tag{3.16}$$

Now, after solving the integral, we have

$$R_{s,k} = \sum_{i=s}^{k} \sum_{j=0}^{i} \binom{k}{i} \binom{i}{j} (-1)^j \frac{\eta_2}{\left(\left(j + k - i\right) \eta_1 + \eta_2\right)}.$$

If $k = s = 1$, then the stress-strength model is simplified to the form stated as

$$R_{1,1} = \sum_{j=0}^{1} \binom{1}{j} (-1)^j \frac{\eta_2}{j\eta_{1+}\eta_2}.$$

Therefore, we have $R_{1,1} = \eta_1/(\eta_{1+}\eta_2)$. Notice that we consider the well known value in the identically distributed case where $\eta_1 = \eta_2$, is $R_{1,1} = 0.5$. It does mean that strength and stress are equal in intensity.

4. Moments and associated indicators

4.1 Moments

Moments are relevant for statistical analysis, typically in implementations. The particular parameters which may be utilized to determine the behavior of a homogeneous dataset are named moments. For the UG model, we obtain the kth ordinary moment $\acute{\rho}_k$ given by

$$\acute{\rho}_k = \mathrm{E}\left(Y^k\right) = \int_0^1 y^k \mathrm{d}F\left(y; b, \eta\right); k = 1, \ldots \tag{4.17}$$

First, some notation are provided. The upper incomplete gamma function symbolized by $\Gamma\left(\varrho, x\right)$ is defined as

$$\Gamma\left(\varrho, x\right) = \int_x^\infty u^{\varrho-1} \exp(-u)\mathrm{d}u;\ x > 0, \varrho \in \mathrm{R}. \tag{4.18}$$

Furthermore, the exponential integral function can be specified regarding the upper incomplete gamma function stated as (Olver et al., 2018)

$$E_\hbar\left(X\right) = \int_1^\infty t^{-\hbar} \exp(-tx)\mathrm{d}t, x > 0, \hbar \in \mathrm{R}, \quad E_\hbar\left(X\right) = x^{\hbar-1}\Gamma\left(1-\hbar, x\right), \hbar, x \in \mathrm{R}. \tag{4.19}$$

The given results presents an equation for the kth ordinary moment $\acute{\rho}_k$ of Y based on the exponential integral function and gamma function. For b, $\eta > 0$, the kth ordinary moment of Y can be written as

$$\acute{\rho}_k\left(y; b, \eta\right) = \eta \exp(\eta) E_{\frac{k}{b}}\left(\eta\right) = \eta^{\frac{k}{b}} \exp(\eta)\Gamma\left(1 - k/b, \eta\right).$$

Specifically, mean of Y is $\acute{\rho}_1 = \mathrm{E}(Y) = \eta^{\frac{1}{b}} \exp(\eta)\Gamma\left(1 - 1/b, \eta\right)$ and facilitate the equation of variance is $\mathrm{Var}(Y) = \acute{\rho}_2 - \left(\acute{\rho}_1\right)^2 = \eta^{\frac{2}{b}} \exp(\eta)\left(\Gamma\left(1 - 2/b, \eta\right) - \exp(\eta)\left(\Gamma\left(1 - 1/b, \eta\right)\right)^2\right)$.

4.2 Indicators and functions associated with moments

In model characterization, the moment generating function (MGF) is widely used. It is easy to express the MGF of the UG model as

$$M\left(y; b, \eta\right) = \sum_{\varsigma=0}^{\infty} \frac{t^\varsigma}{\varsigma!} \acute{\rho}_k\left(y; b, \eta\right) = \sum_{\varsigma=0}^{\infty} \frac{t^\varsigma}{\varsigma!} \eta^{\frac{k}{b}} \exp(\eta)\Gamma\left(1 - k/b, \eta\right).$$

The characteristic function (CF) of Y can be assessed as

$$\Phi\left(\tau y;b,\eta\right)=\int_0^1\exp(i\tau y)\mathrm{d}F\left(y;b,\eta\right)=\sum_{\upsilon=0}^{\infty}\frac{(i\tau)^{\upsilon}}{\upsilon!}\eta^{\frac{k}{b}}\exp(\eta)\Gamma\left(1-k/b,\eta\right).$$

The factorial generating function is obtained as

$$\begin{aligned}F_y\left(\tau y;b,\eta\right)&=\int_0^{\infty}\exp(\log\left(1+\tau\right)^y)\mathrm{d}F\left(y;b,\eta\right),\\&=\sum_{\upsilon=0}^{\infty}\frac{\left(\log\left(1+\tau\right)\right)^{\upsilon}}{\upsilon!}\eta^{\frac{k}{b}}\exp(\eta)\Gamma\left(1-k/b,\eta\right).\end{aligned}$$

The incomplete ordinary moments perform a significant role in determining curves of income inequality, such as the Lorenz curve, which is the most renowned inequality curve utilized in the literature, and several income inequality indices are explicitly obtained from this curve based on incomplete moments.

Proposition 4. Let Y be an RV. Then, the kth incomplete moment of Y is given by

$$\acute{\rho}_{Y,k}\left(z\right)=\eta\exp(\eta)z^{-(b-k)}E_{\frac{k}{b}}\left(\eta z^{-b}\right)=\eta^{\frac{k}{b}}\exp(\eta)\Gamma\left(1-k/b,\eta z^{-b}\right). \tag{4.20}$$

4.3 Numerical illustration

As an empirical illustration, for certain parametric values, $Q_1;b,\eta$, Median$;b,\eta$, $Q_3;b,\eta$, $\mathrm{S}_B;b,\eta$, and $\mathrm{K}_M;b,\eta$ of the UG model are given in Table 1. These outcomes indicate that there are important effects of b and η on said measures. It is observed from Table 1 that across various relevant parameter scenarios, an increasing trend is evident for the lower quartile ($Q_1;b,\eta$), median (Median$;b,\eta$), and upper quartile ($Q_3;b,\eta$). Conversely, an opposite trend is noted for skewness ($\mathrm{S}_B;b,\eta$) and kurtosis ($\mathrm{K}_M;b,\eta$) across different patterns of the parameters. We have $\mathrm{S}_{Bb,\eta}>0$. Therefore, the model is right-skewed with small variations for $\mathrm{K}_M;b,\eta$.

Table 1.: Descriptive measures of the UG model for certain values of b and η.

(b,η)	$Q_1;b,\eta$	Median$;b,\eta$	$Q_3;b,\eta$	$\mathrm{S}_B;b,\eta$	$\mathrm{K}_M;b,\eta$
$(0.25,0.50)$	0.16236	0.00495	0.03084	0.67092	2.08971
$(0.35,0.50)$	0.27293	0.02251	0.08333	0.51431	1.57212
$(0.45,0.50)$	0.36423	0.05231	0.14475	0.40729	1.36290
$(0.50,0.50)$	0.40294	0.07026	0.17561	0.36666	1.30340
$(0.55,0.50)$	0.43765	0.08945	0.20570	0.33228	1.26019
$(0.35,0.25)$	0.11214	0.00466	0.02251	0.66783	2.36850
$(0.35,0.35)$	0.18014	0.01030	0.04415	0.60137	1.92782
$(0.35,0.45)$	0.24361	0.01799	0.06969	0.54171	1.66581
$(0.35,0.55)$	0.30058	0.02743	0.09730	0.48841	1.49518
$(0.35,0.65)$	0.35100	0.03829	0.12572	0.44083	1.37736

4.4 Conditional moments

The conditional moments $\mathrm{E}(Y^k|Y > t)$, for $k = 1, 2$, are helpful in interaction with lifetime distributions in predictive inference. The kth conditional moment of the UG model is stated as

$$\mathrm{E}\Big(Y^k|Y > t\Big) = \frac{1}{S(t)}\left(\mathrm{E}\left(Y^k\right) - \int_0^t y^k f(y)\,\mathrm{d}y\right) = \frac{\eta^{\frac{k}{b}}\exp(\eta)\Gamma(1-k/b,\eta) - \acute{\mu}_{Y,k}(t)}{1-\exp\left(\eta\left(1-t^{-b}\right)\right)}.$$

The mean deviations provide important information about the characteristics of a population and can be estimated from the first incomplete moment. Additionally, the extent of dispersion in a dataset can be partially determined by considering all deviations from the median and mean. The mean deviations of Y about mean $\acute{\mu}_1 = \mathrm{E}(Y)$ and about median M_{ed} are stated as $\Phi = 2F(\acute{\mu}_1) - 2\lambda_1\acute{\mu}_1$ and $\Psi = \acute{\mu}_1 - 2\lambda_1 M$, where $\lambda_1(z) = \int_0^z yf(y)\,\mathrm{d}y$ and $F(\acute{\mu}_1)$ is specified in Equation (2.3).

5. Uncertainty measures and residual life function

5.1 Information generating function

The differentiation of the information generating function (IGF) at zero or one enables us to extract measures of information that are otherwise difficult to describe and calculate. For the UG model, the IGF of Y is given by

$$\tilde{\mathrm{I}}_\psi(f) = \mathrm{E}\left(f^{\psi-1}(y;b,\eta)\right) = \int_0^1 f^{\psi}(y;b,\eta)\,\mathrm{d}y = \int_0^1 \left(b\eta y^{-b-1}\exp\left(\eta\left(1-y^{-b}\right)\right)\right)^{\psi}\mathrm{d}y. \tag{5.21}$$

Put $t = \psi\eta y^{-b} \Rightarrow \mathrm{d}t = -1/\left(bt^{1+1/b}(\eta\psi)^{-1/b}\right)\mathrm{d}t$ in Equation (5.21) and after a some simplification $\tilde{\mathrm{I}}_\psi(f)$ is reduced to

$$\tilde{\mathrm{I}}_\psi(f) = b^{\psi-1}\eta^{\frac{1-\psi}{b}}\psi^{(1-\psi)\frac{1}{b}-\psi}\exp(\psi\eta)\Gamma\left(\psi+\frac{\psi-1}{b},\eta\psi\right).$$

5.2 Entropy measures

Entropy is a significant concept in various fields such as thermodynamics, communication, information theory, topological dynamics, statistical mechanics, and measure-preserving dynamical systems. It serves as a measure for different attributes like chaos, work-impossible energy, uncertainty, randomness, complexity, etc. However, note that there are multiple concepts of entropy, and they may not be universally applicable to all situations.

The Shannon entropy is defined as

$$\begin{aligned} S(Y) &= \mathrm{E}\left(-\log\left(f(Y)\right)\right) \\ &= -\int_0^1 \log\left((y;b,\eta)\right) f(y;b,\eta)\,\mathrm{d}y, \qquad (5.22) \\ &= -\left(\log\left(b\eta\exp(\eta)\right) - b\eta(b+1)\int_0^1 \log(y)\,y^{-b-1}\exp\left(\eta\left(1-y^{-b}\right)\right)\mathrm{d}y\right. \\ &\quad \left. -b\eta^2\int_0^1 y^{-b}y^{-b-1}\exp\left(\eta\left(1-y^{-b}\right)\right)\right)\mathrm{d}y. \end{aligned}$$

By using the transformation $t=y^{-b}$ and after some simplification, we have

$$S\left(Y\right)=-\log\left(b\eta\exp(\eta)\right)+\left(\frac{b+1}{b}\right)E_1\left(\eta\right)-\left(\exp(-\eta)-\frac{\exp(-\eta)}{\eta^2}\right).$$

Then, we have

$$S\left(Y\right)=\frac{1}{\eta}-\log(b)-\log(\eta)-\left(\frac{b+1}{b}\right)E_1\left(\eta\right).$$

The Renyi entropy $\tilde{I}_\delta\left(Y\right)$ is defined as

$$\tilde{I}_\delta\left(Y\right)=\left(\frac{1}{1-\delta}\right)\log\int_0^\infty f^\delta\left(y;b,\eta\right)\mathrm{d}y;\ \ \delta\neq1,\delta>0,$$

where

$$f^\delta\left(y;b,\eta\right)=\left(b\eta y^{-b-1}\exp\left(\eta\left(1-y^{-b}\right)\right)\right)^\delta.$$

Using the information mentioned above, we obtain

$$\int_0^\infty f^\delta\left(y;b,\eta\right)\mathrm{d}y=b^{\delta-1}\eta^{\frac{1-\delta}{b}}\delta^{(1-\delta)\frac{1}{b}-\delta}\exp(\delta\eta)\int_0^\infty z^{\frac{\delta}{b}(b+1)-\frac{1}{b}-1}\exp(-z)\mathrm{d}z.$$

Then, we have

$$\tilde{I}_\delta\left(Y\right)=\left(\frac{1}{1-\delta}\right)\log\left(b^{\delta-1}\eta^{\frac{1-\delta}{b}}\delta^{(1-\delta)\frac{1}{b}-\delta}\exp(\delta\eta)\Gamma\left(\delta+\frac{\delta-1}{b},\eta\delta\right)\right).$$

It is necessary to note that Shannon entropy $S\left(Y\right)$ is given in Equation (5.22) is accomplished as a unique case of Renyi entropy $\tilde{I}_\delta\left(Y\right)$ for $\delta\to1$.

The Verma entropy $V_{\delta,\beta}\left(Y\right)$ is

$$V_{\delta,\beta}\left(Y\right)=\left(\frac{1}{\delta-\beta}\right)\log\int_{-\infty}^\infty f^{\delta+\beta-1}\left(y;b,\eta\right)\mathrm{d}y;\ \ \beta-1<\delta<\beta,\beta\geq1,\delta\neq\beta,\qquad(5.23)$$

where

$$f^{\delta+\beta-1}\left(y;b,\eta\right)=\left(b\eta y^{-b-1}\exp\left(\eta\left(1-y^{-b}\right)\right)\right)^{\delta+\beta-1}.$$

It is significant to noted that, for $\beta\to1$, in Equation (5.23), the Renyi entropy is attained. Furthermore, for $\beta\to1$ and $\delta\to1$, in Equation (5.23), then it reduces to Shannon entropy. Utilizing above mentioned information, we have

$$V_{\delta,\beta}\left(Y\right)=\left(\frac{1}{\delta-\beta}\right)\log\left(b^{\delta+\beta-2}\eta^{\frac{2-\delta-\beta}{b}}\left(\delta+\beta-1\right)^{(2-\delta-\beta)\frac{1}{b}-(\delta+\beta-1)}\exp\left(\left(\delta+\beta-1\right)\eta\right)\right.$$
$$\left.\Gamma\left(\left(\delta+\beta-1\right)\left(1+\frac{1}{b}\right)-\frac{1}{b},\eta\left(\delta+\beta-1\right)\right)\right).$$

The Tsallis entropy $T_\delta(Y)$ is given by

$$T_\delta(Y) = \frac{1}{\delta-1}\left(1 - \int_0^\infty f^\delta(y;b,\eta)\,\mathrm{d}y\right),\ \delta \neq 1.$$

Using the above transformation, we have

$$T_\delta(Y) = \frac{1}{\delta-1}\left(1 - b^{\delta-1}\eta^{\frac{1-\delta}{b}}\delta^{(1-\delta)\frac{1}{b}-\delta}\exp(\delta\eta)\Gamma\left(\delta+\frac{\delta-1}{b},\eta\delta\right)\right).$$

The classical Shannon entropy has been generalized in several directions, one of which is the generalized entropy δ_1 introduced by Mathai et al. (2013) and defined as

$$\tilde{I}_{\delta_1}(Y) = \left(\frac{1}{\delta_1-1}\right)\int_0^\infty f^{2-\delta_1}(y;b,\eta)\,\mathrm{d}y - 1,\ \delta_1 \neq 1.$$

Relevant assertions to $f^{2-\delta_1}$ gives the expression stated as $f^{2-\delta_1}(y;b,\eta) = \left(b\eta y^{-b-1}\exp\left(\eta\left(1-y^{-b}\right)\right)\right)^{2-\delta_1}$. Consequently, the above integral is formulated as

$$\tilde{I}_{\delta_1}(Y) = \frac{1}{1-\delta_1}\Big(b^{1-\delta_1}\eta^{\frac{\delta_1-1}{b}}(2-\delta_1)^{(\delta_1-1)\frac{1}{b}-(2-\delta_1)}\exp\left((2-\delta_1)\eta\right) \times \Gamma\Big((2-\delta_1)\left(1+\frac{1}{b}\right)-\frac{1}{b},\eta(2-\delta_1)\Big)-1\Big).$$

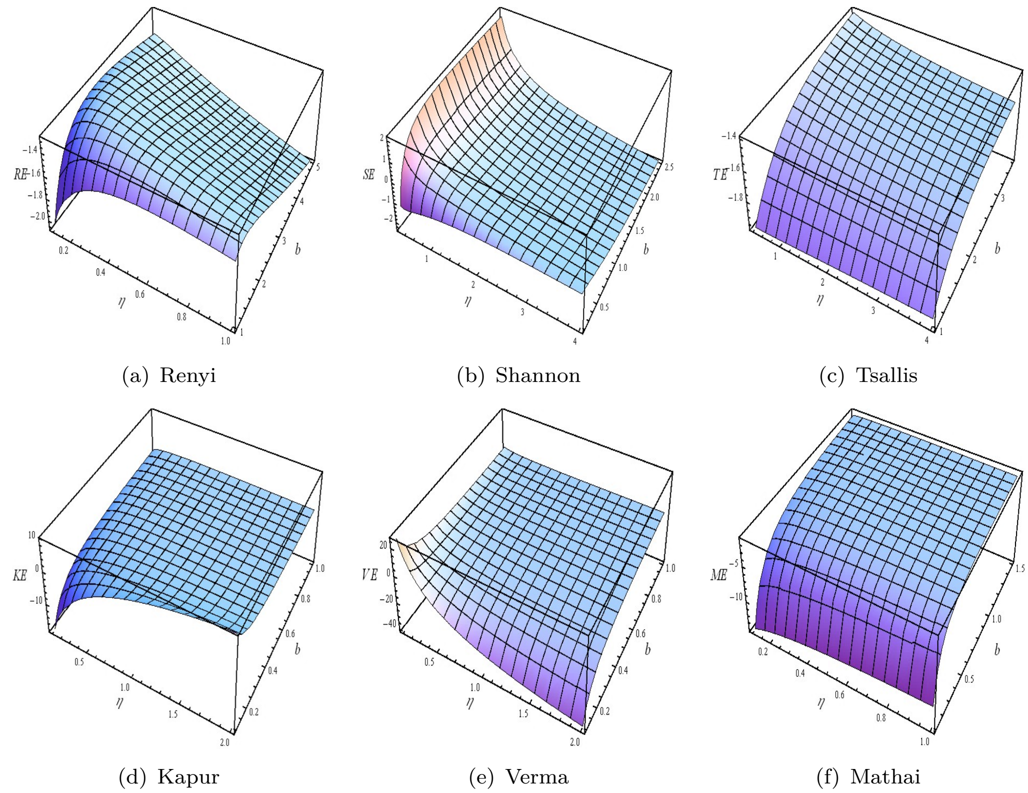
The Kapur entropy $\tilde{I}_{\alpha,\beta}(Y)$ is defined as

$$\begin{aligned}\tilde{I}_{\alpha,\beta}(Y) &= \left(\frac{1}{\beta-\alpha}\right)\log\left(\frac{\int_0^\infty f^\alpha(y)\,\mathrm{d}y}{\int_0^\infty f^\beta(y)\,\mathrm{d}y}\right), \\ &= \frac{1}{\beta-\alpha}\left(\log\left(\int_0^\infty f^\alpha(y)\,dy\right)-\log\left(\int_0^\infty f^\beta(y)\,\mathrm{d}y\right)\right),\end{aligned} \tag{5.24}$$

Relevant assertions to f^δ used in Equation (5.24) stated the expression given by

$$\tilde{I}_{\alpha,\beta}(Y) = \frac{1}{\beta-\alpha}\log\left(\frac{b^{\alpha-1}\eta^{\frac{1-\alpha}{b}}\delta^{(1-\alpha)\frac{1}{b}-\alpha}\exp(\alpha\eta)\Gamma\left(\alpha\left(1+\frac{1}{b}\right)-\frac{1}{b},\eta\alpha\right)}{b^{\beta-1}\eta^{\frac{1-\delta}{b}}\delta^{(1-\delta)\frac{1}{b}-\delta}\exp(\beta\eta)\Gamma\left(\beta\left(1+\frac{1}{b}\right)-\frac{1}{b},\eta\beta\right)}\right).$$

3D behavior of entropies (Shannon, Renyi, Mathai, Tsallis, Kapur and Verma entropy) of the UG distribution are plotted in Figures 8(a)-(f). The information in Table 2 outlines the variations in Shannon, Renyi, Mathai, Tsallis, Kapur, and Verma entropy with different parameter values in the UG distribution, while holding other parameters constant. Notably, Shannon, Mathai, and Verma entropies presents an increasing trend with the parameter b, whereas Renyi, Tsallis, and Kapur entropies depict a decreasing trend by keeping other parameter fixed (η; δ_1; δ; β). For elevated values of η, Shannon, Renyi, Mathai, Tsallis, and Kapur entropies rise, while Verma entropy declines with the other parameters maintained. Moreover, an increase in δ_1 leads to constancy in Shannon, Renyi, Kapur, Tsallis, and Verma entropies, while Mathai entropy experiences a decreasing trend. On the contrary, changes in δ result in Shannon entropy exhibiting a constant behavior, while other entropies display mixed patterns of increase and decrease.



(a) Renyi (b) Shannon (c) Tsallis

(d) Kapur (e) Verma (f) Mathai

Figure 8.: Behavior of various entropies of the UG distribution.

Table 2.: The values of the Shannon, Renyi, Mathai, Tsallis, Kapur and Verma entropy of the UG distribution for some parameter values.

					Entropies					
b	η	δ_1	δ	β	Shanon	Renyi	Mathai	Tsallis	Kapur	Verma
0.2	0.5	1.1	2.5	3	−1.23488	3.41897	−6.27843	0.66272	3.14144	−48.6853
0.6	0.5	1.1	2.5	3	0.74288	1.58266	−2.17344	0.60459	1.28606	−22.0735
1.0	0.5	1.1	2.5	3	0.84733	0.94298	−1.53368	0.50463	0.59889	−14.7177
2.0	0.5	1.1	2.5	3	0.61563	0.18146	−1.34429	0.15886	−0.23716	−7.06456
2.0	0.4	1.1	2.5	3	1.15140	0.20661	−1.31290	0.17766	−0.22597	−7.15680
2.0	0.6	1.1	2.5	3	0.24245	0.14452	−1.37855	0.12993	−0.26731	−6.85219
2.0	0.8	1.1	2.5	3	−0.25687	0.05571	−1.44841	0.05344	−0.35547	−6.24762
0.6	0.6	0.2	4.5	5	0.48050	3.32872	0.54551	0.28571	3.06992	−74.5745
0.6	0.6	0.5	4.5	5	0.48050	3.32872	0.08206	0.28571	3.06992	−74.5745
0.6	0.6	1.3	4.5	5	0.48050	3.32872	−2.63598	0.28571	3.06992	−74.5745
0.6	0.6	1.5	4.1	5	0.48050	3.08392	−2.46861	0.32256	2.82143	−38.2444
0.6	0.6	1.5	4.5	5	0.48050	3.32872	−2.46861	0.28571	3.06992	−74.5745
0.6	0.6	1.5	4.9	5	0.48050	3.55110	−2.46861	0.25641	3.29715	−402.085

5.3 Residual life function with a measure of reliability

RVs of residual life and inverted residual life are widely practiced in risk investigation. Hence, in connection with the UG distribution, we explore some associated statistical features, like mean, SF and variance.

The residual life is explained by conditional RV $R(t) = Y - t; Y > t$, $t \geq 0$, and described as period between the moment t and the moment of failure. The reversed residual life (or time since failure) can also be described as $\ddot{R}(t) = t - Y; Y \leq t$, This refers to the moment elapsed due to the component failure, given that its lifetime $\leq t$ (Das and Nanda , 2013; Tang et al., 1999; Siddiqui and Çağlar , 1994).

The SF of residual lifetime, $R(t)$ namely, with $t \geq 0$ and $y > 0$, for the UG model is given by

$$S_{R(t)}(y) = \frac{S(y+t)}{S(t)} = \frac{\left(1 - \exp\left(\eta\left(1 - (y+t)^{-b}\right)\right)\right)}{\left(1 - \exp\left(\eta\left(1 - t^{-b}\right)\right)\right)}.$$

Thus, the PDF of $R(t)$ simplifies to the expression stated as

$$f_{R(t)}(y; \eta, \delta, \upsilon) = \frac{b\eta (y+t)^{-b-1} \exp\left(\eta\left(1 - (y+t)^{-b}\right)\right)}{\left(1 - \exp\left(\eta\left(1 - t^{-b}\right)\right)\right)}.$$

Therefore, the HRF of $R(t)$ is provided by

$$h_{R(t)}(y) = \frac{b\eta (y+t)^{-b-1} \exp\left(\eta\left(1 - (y+t)^{-b}\right)\right)}{\left(1 - \exp\left(\eta\left(1 - (y+t)^{-b}\right)\right)\right)}.$$

The mean residual life function (M_{RL}) has many application, like in maintenance, insurance and quality control of products, social studies and economics. For the UG distribution, we can represent its mean residual life as

$$\begin{aligned}
\Lambda(t) = \mathrm{E}(R(t)) &= \frac{1}{(1 - F(t))} \int_t^\infty y f(y)\,\mathrm{d}y - t, \\
&= \frac{1}{(1 - F(t))} \left(\mathrm{E}(Y) - \acute{\rho}_{Y,1}(z)\right) - t, \qquad t \geq 0,
\end{aligned}$$

where $f(y), F(y)$ are specified in Equation (2.2), (2.3) and $\mathrm{E}(Y) = \eta^{\frac{1}{b}} \exp(\eta)\Gamma(1 - 1/b, \eta)$. The variance residual life (V_{RL}) is another measure of concern that has increased attention in latest years (Khorashadizadeh et al., 2013; Gupta, 2006), which is defined as

$$\begin{aligned}
V_{\mathrm{RL}} = V(R(t)) &= \frac{2}{S(t)} \int_t^\infty y S(y)\,\mathrm{d}y - 2t\Lambda(t) - \Lambda^2(t), \\
&= \frac{1}{S(t)} \left(\mathrm{E}(Y^2) - \acute{\rho}_{Y,2}(z)\right) - t^2 - 2t\Lambda(t) - \Lambda^2(t),
\end{aligned}$$

where $\mathrm{E}\left(Y^2\right) = \eta^{\frac{2}{b}} \exp(\eta)\Gamma(1 - 2/b, \eta)$ and $\acute{\rho}_{Y,2}(z)$ specified in Equation (4.20) by setting $r = 2$.

The SF of the reversed residual lifetime, $\dddot{R}(t)$ say, with $0 \leq y < t$, for the UG model is established as

$$S_{\dddot{R}(t)}(y) = \frac{F(t-y)}{F(t)} = \frac{\left(\exp\left(\eta\left(1 - (t-y)^{-b}\right)\right)\right)}{\left(\exp\left(\eta\left(1 - (t)^{-b}\right)\right)\right)}.$$

Then, the $\ddot{R}(t)$ PDF becomes formulated as

$$f_{\ddot{R}(t)}(y;\eta,\delta,\upsilon)=\frac{b\eta\,(t-y)^{-b-1}\exp\left(\eta\left(1-(t-y)^{-b}\right)\right)}{\left(\exp\left(\eta\left(1-t^{-b}\right)\right)\right)}.$$

Hence, the HRF of $\ddot{R}(t)$ is reduced to the formula given by

$$h_{\ddot{R}(t)}(y)=\frac{b\eta\,(t-y)^{-b-1}\exp\left(\eta\left(1-(t-y)^{-b}\right)\right)}{\left(\exp\left(\eta\left(1-(t-y)^{-b}\right)\right)\right)}.$$

The mean and variance of $\ddot{R}(t)$ are presented as

$$\dddot{\Lambda}(t)=\mathrm{E}\left(\ddot{R}(t)\right)=t-\frac{1}{F(t)}\int_0^t yf(y)\,\mathrm{d}y=t-\frac{1}{F(t)}\left(\acute{\rho}_{t,1}(z)\right),\ 0<y<t;$$

$$\begin{aligned}\mathrm{V}_{\ddot{R}L}&=\mathrm{Var}\left(\ddot{R}(t)\right)=2t\dddot{\Lambda}(t)-\dddot{\Lambda}^2(t)-\frac{2}{F(t)}\int_0^t yF(y)\,\mathrm{d}y,\\&=2t\dddot{\Lambda}(t)-\dddot{\Lambda}^2(t)-t^2+\frac{1}{F(t)}\left(\acute{\rho}_{t,2}(z)\right),\end{aligned}$$

in which $F(t)$, $f(y)$ and $\acute{\rho}_{t,2}(z)$ can be identified from Equations (2.3), (2.2) and (4.20) by setting $r=2$, respectively.

The Bonferroni and Lorenz curves are income inequality measures that are commonly applicable to certain other fields including reliability, demography, medicine and insurance and medicine. The Bonferroni curve $\mathrm{B}_{F(y)}$ of Y is stated as

$$\mathrm{B}_{F(y)}=\frac{1}{\mathrm{E}(Y)F(y)}\int_0^y yf(y)\,\mathrm{d}y=\frac{\acute{\rho}_{Y,1}(z)}{\mathrm{E}(Y)F(y)}.$$

Groves-Kirkby et al. (2009) highlighted the significance of the Lorenz curve for applications in various scientific fields. The Lorenz curve $\mathrm{L}_{F(y)}$ of Y is expressed as

$$\mathrm{L}_{F(y)}=\frac{1}{\mathrm{E}(Y)}\int_0^y yf(y)\,\mathrm{d}y=\frac{\acute{\rho}_{Y,1}(z)}{\mathrm{E}(Y)}.$$

6. Estimation, simulation, and real data analysis

6.1 Computational setting

Using the `R` software and its features `optimum` and `MaxLik`; the `Ox` program and its subroutine `MaxBFGS`; the `Matlab` software with tool `log_lik`; and the `SAS` software with its procedure `PROC NLMIXED`); the parameters of the UG model can be estimated from the corresponding log-likelihood function based on sample data. Furthermore, certain goodness-of-fit statistics are used for model fitting and selection with the distributions considered.

6.2 Maximum likelihood estimation

The maximum likelihood (ML) estimated are obtained by optimizing the corresponding equation in accordance with η and b.

The ML. estimates are the maximum of log-likelihood function defined as $l_{\mathbf{y};\eta,b} = \log L(\mathbf{y};\eta,b)$. For a given data $y_1,\ldots,y_n$, the log-likelihood function for the UG model is given by

$$L(\mathbf{y};\eta,b) = \prod_{i=1}^{n} b\eta y^{-b-1}\exp\Big(\eta\Big(1-y^{-b}\Big)\Big),$$
$$l_{\mathbf{y};\eta,b} = n\log(b) + n\log(\eta) - (b+1)\sum_{i=1}^{n}\log(y_i) + \eta\Big(n - \sum_{i=1}^{n} y_i^{-b}\Big).$$

To obtain the ML estimates, we get the components of the parameter vector $\Lambda_{\eta b}$=$(\Lambda_\eta,\Lambda_b)^\tau$, and set them zero. Such components are given by

$$\Lambda_b = \frac{\partial l_{\mathbf{y};\eta,b}}{\partial l_{\mathbf{y};b}} = \frac{n}{b} - \sum_{i=1}^{n}\log(y_i) + \eta\sum_{i=1}^{n} y_i^{-b}\log(y_i)\,.$$
$$\Lambda_\eta = \frac{\partial l_{\mathbf{y};\eta,b}}{\partial l_{\mathbf{y};\eta}} = n + \frac{n}{\eta} - \sum_{i=1}^{n} y_i^{-b}.$$

The ML estimate $(\widehat{\eta},\widehat{b})$ of (η,b) is obtained setting $\Lambda_\eta = \Lambda_b = 0$ and solving them simultaneously. It indicates that the ML estimate of η can be obtained directly as $n/(\sum_{i=1}^{n} y_i^{-b} - n)$. To estimate confidence intervals for parameters, we need the 2×2 information matrix $J(\Lambda_{\eta b}) = J_{\eta b}(\Lambda_{\eta b})$. The asymptotic distribution $(\widehat{\Lambda}_{\eta b} - \Lambda_{\eta b})$ is $\mathrm{N}_2\left(\mathbf{0},\Delta(\Lambda_{\eta b})^{-1}\right)$, where $\Delta(\Lambda_{\eta b}) = \mathrm{E}(J(\Lambda_{\eta b}))$. The approximate multivariate normal distribution, $\mathrm{N}_2\left(\mathbf{0},J(\Lambda_{\eta b})^{-1}\right)$ say, where $J(\Lambda_{\eta b})^{-1}$ is the inverse of information matrix at $\Lambda_{\eta b} = \widehat{\Lambda}_{\eta b}$, can be implemented upon usual regularity conditions to develop approximate confidence intervals for the model parameters.

6.3 Simulation

The statistical properties of the ML estimators for the UG distribution are difficult to compare from a theoretical perspective. However, we suggest a simulation analysis using their mean square errors (MSEs) as a standard for various sample sizes. The study considered various parameters sets: $(0.5,0.7)\,,(1.5,2.5)\,,(1.0,3.0)\,,$and $(3.0,1.5)$. We also consider the following sample sizes: $n = 30, 70, 100, 200, 300$ and 500. All simulations were run using the `R` programming language. The performance of these estimators is assessed using empirical meand and MSEs, which are calculated using the `optim` function and the Nelder-Mead method ofl `R`. To assess the effectiveness of the estimators, Algorithm 1 was used.

Algorithm *1* Algorithm

1: Generate pseudo-random numbers of size n using the UG distribution.
2: Find the ML estimates $\widehat{b}_{\mathrm{ML}}$ and $\widehat{\eta}_{\mathrm{ML}}$ for the UG model.
3: Perform steps 1 and 2 $M = 5000$ times.
4: Use $\widehat{b}_{\mathrm{ML}}$ and $\widehat{\eta}_{\mathrm{ML}}$ for determinate the estimates, and MSEs of the estimations.

Based on the empirical mean and MSEs of the ML estimators, a validation study is conducted. Various set sizes, and distinct parameters levels are all considered when utilizing Monte Carlo simulation.

For each simulated scenario, we use

$$\text{Mean}_\theta(n) = \frac{1}{M}\sum_{j=1}^{M}\tilde{\Theta}_j, \quad \text{MSE}_\theta(n) = \frac{1}{M}\sum_{j=1}^{M}\left(\tilde{\Theta}_j - \Theta\right)^2.$$

The empirical mean and MSE are presented in Tables 3 and 4. The results illustrate that the average estimates are closer to the true values of the parameters as the sample size increases. Further, the MSEs for all estimates decrease with the increase in sample size. The method demonstrates the consistency property of the ML estimators. We conclude that the ML approach performs well in estimating the parameters of the UG distribution.

Table 3.: Empirical mean and MSEs of the ML estimators for the UG distribution with values of $(b, \eta) \in \{(0.5, 0.7), (1.5, 2.5)\}$.

n	Parameter	Initial Value	MLE	MSE	Initial value	MLE	MSE
30	b	0.5	0.58555	0.04278	1.5	2.01564	1.58644
	η	0.7	1.21662	1.73062	2.5	2.74294	6.64286
70	b	0.5	0.51695	0.01752	1.5	1.77301	0.62046
	η	0.7	0.80272	0.28841	2.5	2.65420	6.13443
100	b	0.5	0.51511	0.01083	1.5	1.69771	0.46644
	η	0.7	0.73911	0.12824	2.5	2.62053	3.79372
200	b	0.5	0.50637	0.00496	1.5	1.60774	0.22182
	η	0.7	0.71752	0.03497	2.5	2.59669	1.45851
300	b	0.5	0.50492	0.00354	1.5	1.56151	0.14441
	η	0.7	0.70977	0.02451	2.5	2.56815	0.87391
500	b	0.5	0.50486	0.002107	1.5	1.50541	0.07173
	η	0.7	0.70169	0.014406	2.5	2.50602	0.44074

Table 4.: Empirical mean and MSEs of the ML estimators for the UG distribution with values of $(b, \eta) \in \{(1.0, 3.0), (3.0, 1.5)$.

n	Parameters	Initial Value	MLE	MSE	Initial Value	MLE	MSE
30	b	1.0	1.40271	0.99856	3.0	3.67576	3.62236
	η	3.0	4.03190	9.05440	1.5	2.02742	8.38644
70	b	1.0	1.17146	0.36151	3.0	3.28572	1.29957
	η	3.0	3.75358	8.87920	1.5	1.66935	1.56111
100	b	1.0	1.11537	0.23260	3.0	3.22973	0.94178
	η	3.0	3.58433	8.20469	1.5	1.57504	0.66874
200	b	1.0	1.05919	0.11352	3.0	3.10632	0.42723
	η	3.0	3.28323	3.42921	1.5	1.54323	0.27760
300	b	1.0	1.03126	0.07672	3.0	3.05927	0.26288
	η	3.0	3.24122	2.03550	1.5	1.53031	0.16035
500	b	1.0	1.02841	0.04649	3.0	3.05351	0.16284
	η	3.0	3.08923	0.83264	1.5	1.50436	0.09279

6.4 Real data applications

We examine two real-life datasets for illustrative purposes to see if our distribution provides a better fit for data than some other standard distributions. The performance of the UG distribution is compared with the Kumaraswamy (KSW), size-biased Kumaraswamy (SBKSW), beta, transmuted Kumaraswamy (TKSW) and exponentiated generalized Kumaraswamy (EG-KSW) models via suitable measures such as the Akaike information criterion (AIC), Hannan-Quinn information criterion (HQIC), Bayesian information criterion (BIC) and consistent Akaike information criterion (CAIC). Further, formal goodness-of-fit measures as Cramer-von Mises (W^*), Anderson-Darling (A^*), and log-likelihood ($\widehat{l}$), are used to check which distribution fits better these datasets. Using these goodness of fit measures, the results are presented in Tables 6-9. As expected, the lower of these criteria suggest the model with a better fit. Likewise, Kolmogorov-Smirnov (KS) and p-values are reported. We notice that with largest p-values, the UG model does have smallest statistics then providing the best fit among the compared distributions.

Data 1: Flood data. The flood data of 20 observations is used here. The data were also examined in Dumonceaux et al. (1973) and are given by

0.265, 0.297, 0.392, 0.3235, 0.269, 0.402, 0.315, 0.338, 0.379, 0.654, 0.418, 0.379, 0.412, 0.423, 0.416, 0.484, 0.449, 0.613, 0.494, 0.74.

Data 2: Strengths of 1.5 cm glass fibers. The following data reflects the strengths of 1.5 cm glass fibers, initially collected by employees at the UK National Physical Laboratory. The data are given by

0.13, 0.17, 0.16, 0.20, 0.14, 0.15, 0.11, 0.13, 0.15, 0.12, 0.15, 0.12, 0.12, 0.16, 0.21, 0.23, 0.20, 0.16, 0.12, 0.32, 0.10, 0.33, 0.36, 0.33, 0.38, 0.26, 0.20.

A summary for the data is provided in Table 5 and the boxplot for both datasets are plotted in Figure 9. Figures 10-11 present the probability-probability (PP) plots of the fitted models. For each model, we estimate unknown parameters by paradigm of ML estimation. Tables 10 and 11 show the ML estimates with their respective standard errors (SEs) of above distributions using `R`.

Table 5.: Descriptive measures for datasets 1 and 2.

Dataset	Min	1st Quartile	Median	Mean	3rd Quartile	Max
1	0.2650	0.3344	0.4070	0.4231	0.4577	0.7400
2	0.100	0.130	0.160	0.193	0.220	0.380

Table 6.: Values of the performance indicators for dataset 1.

Distribution	W^*	A^*	KS value	KS p-value
UG(η, b)	0.05297	0.30130	0.14682	0.8017
KSW(α, β)	0.16581	0.97220	0.21093	0.3358
SBKSW(λ, δ)	0.04631	0.30000	0.96493	< 0.0001
Beta(α, β)	0.07313	0.44410	0.99997	< 0.0001
TKSW(α, β, θ)	0.14093	0.84098	0.19299	0.4457
EG-KSW$(\alpha, \beta, \lambda, \kappa)$	0.06740	0.41334	0.15615	0.7140

Table 7.: Values of the performance indicators for dataset 2.

Distribution	W^*	A^*	KS value	KS p-value
UG (η, b)	0.05394	0.39496	0.12328	0.8064
KSW(α, β)	0.23762	1.43931	0.19094	0.2785
SBKSW(λ, δ)	0.07270	0.53266	0.99514	2.2×10^{-16}
Beta(α, β)	0.06905	0.51241	0.99897	2.2×10^{-16}
EG-KSW$(\alpha, \beta, \lambda, \kappa)$	0.11309	0.74610	0.16275	0.4719

Table 8.: Performance measures for all models for dataset 1.

Distribution	AIC	CAIC	BIC	HQIC	$-\widehat{l}$
UG (η, b)	−28.71941	−28.01353	−26.72795	−28.33066	−16.35971
KSW(α, β)	−21.73238	−21.0265	−19.74092	−21.34363	−12.86619
SBKSW(λ, δ)	−22.45675	−21.75086	−20.46528	−22.06799	−13.22837
TKSW(α, β, θ)	−21.04188	−19.54188	−18.05468	−20.45874	−13.52094
Beta(α, β)	−24.12447	−23.41859	−22.13301	−23.73571	−14.06223
EG-KSW$(\alpha, \beta, \lambda, \kappa)$	−23.42618	−20.75951	−19.44325	−22.64867	−15.71309

Table 9.: Performance measures for all models for dataset 2.

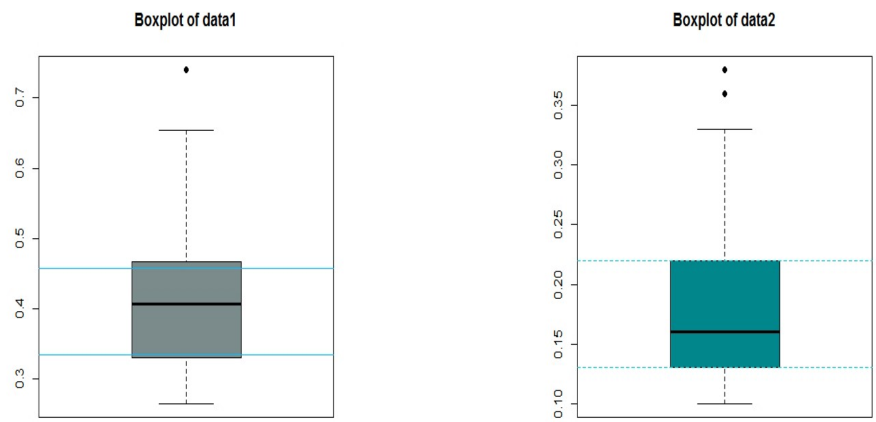
Distribution	AIC	CAIC	BIC	HQIC	$-\widehat{l}$
UG(η, b)	−67.0739	−66.5739	−64.48223	−66.30326	−35.53695
KSW(α, β)	−57.34612	−56.84612	−54.75445	−56.57548	−30.67306
SBKSW(λ, δ)	−58.56875	−58.06875	−55.97707	−57.79811	−31.28437
Beta(α, β)	−60.54254	−60.04254	−57.95086	−59.7719	−32.27127
EG-KSW$(\alpha, \beta, \lambda, \kappa)$	−60.58179	−58.7636	−55.39844	−59.04051	−34.29089

Table 10.: ML estimates with the corresponding SEs of real dataset 1.

Distribution	Estimate			
UG(η, b)	4.12466	0.01500	-	-
	0.74445	0.01305	-	-
KSW(α, β)	3.36326	11.7886	-	-
	0.60328	5.35874	-	-
SBKSW(λ, δ)	2.77847	10.5672	-	-
	0.61510	4.62355	-	-
Beta(α, β)	6.75693	9.11177	-	-
	2.09457	2.85181	-	-
TKSW(α, β, θ)	3.72639	10.9712	0.61397	-
	0.64923	6.04152	0.37517	-
EG-KSW$(\alpha, \beta, \lambda, \kappa)$	0.47562	2.16057	2.16057	98.96214
	0.97436	1.85918	1.85918	552.08653

Table 11.: ML estimates with the corresponding SEs of real dataset 2.

Distribution	Estimate			
UG (η, b)	2.95800	0.00373	-	-
	0.21864	0.00154	-	-
KSW(α, β)	2.49359	43.54502	-	-
	0.36807	23.26493	-	-
SBKSW(λ, δ)	1.95440	30.67920	-	-
	0.36489	15.25790	-	-
Beta(α, β)	5.07424	21.12431	-	-
	1.33911	05.78928	-	-
EG-KSW$(\alpha, \beta, \lambda, \kappa)$	0.48445	02.84163	2.84164	67.16952
	0.22520	03.74947	3.74948	91.90553

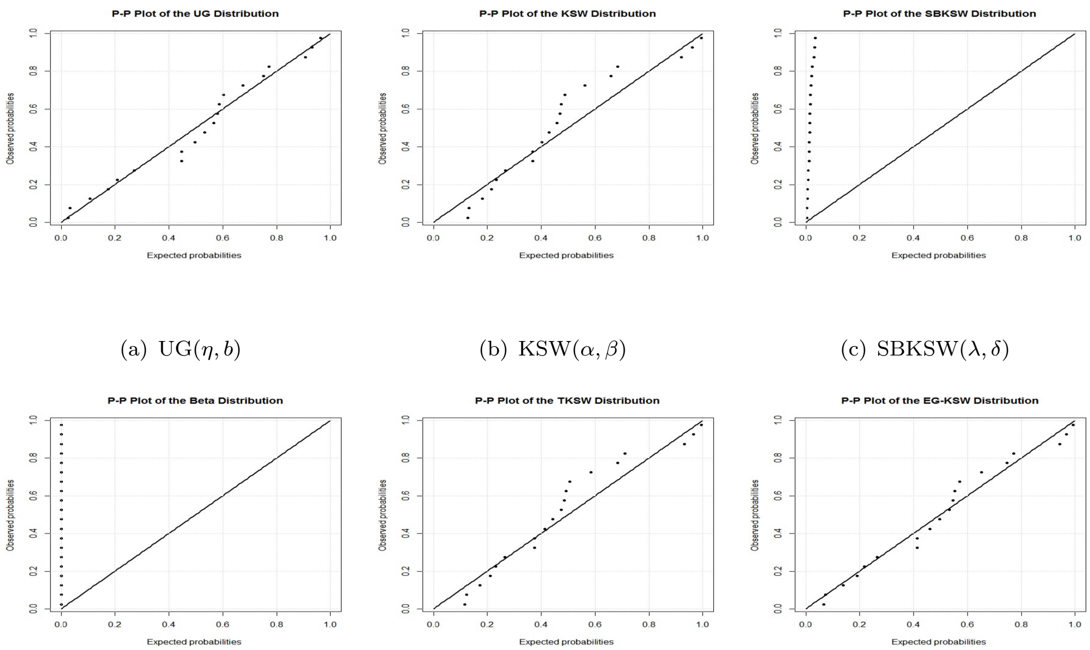


(a) dataset 1 (b) dataset 2

Figure 9.: Boxplots for datasets 1 and 2.

7. Concluding remarks

We implemented the three-parameter lifetime model recognized as the unit-Gompertz distribution. Different mathematical properties were discussed with discussion involving quantile function, stochastic ordering and related measures. We included some figures for probability density function and hazard rate function. The general non-central complete and incomplete moments and residual life function with a certain measure of reliability were also studied. Entropies for uncertainty measures (like Shannon, Tsallis, Renyi, Mathai-Houbold, Kumar and Verma) were obtained, calculated and displayed graphically. For moment generating function, non-moment (complete and incomplete), conditional moments and mean deviations, residual lifetime and reversed residual life functions, we also got explicit expressions for the studied model. To determine variation of maximum likelihood estimates in finite samples, we accessed these estimates via Monte Carlo simulation for complete sample. We tested the effectiveness of the unit-Gompertz model with its five important counterparts by the use of the classical goodness of fit measures and probability plots. These results are consistent with the fact that the current model is very suited for areas of real-life data.



(a) UG(η, b) (b) KSW(α, β) (c) SBKSW(λ, δ)

(d) Beta(α, β) (e) TKSW(α, β, θ) (f) EG-KSW$(\alpha, \beta, \lambda, \kappa)$

Figure 10.: PP plots for dataset 1.

Note that, by solving the linear equation $Q(\zeta, b, \eta) - \mu = 0$ in η, we have $\eta = \log(\zeta)/(1 - \mu^{-b})$, where $Q(\zeta, b, \eta)$ was given in Equation (4.17). In this way, we can parameterize Equations (2.2) and (2.3) as a function of μ, where $0 < \mu < 1$ and $0 < \zeta < 1$ is fixed. In this new parameterization, we can study the effect of covariates on the ζth quantile of the response variable. This strategy has been widely considered in the literature in recent years; see for example, Mazucheli et al. (2021) and the references therein. Quantile regression considering the unit-Gompertz distribution is under development by the authors and will be presented in a future article.

Appendix A: Proof of characterization based on HRF

Proof of Proposition 1

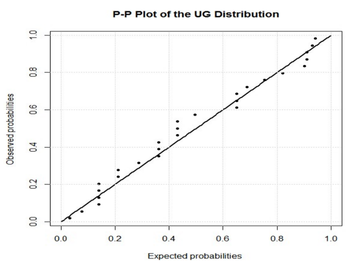
If Y follows Equation (2.2), then obviously Equation (2.9) holds. If $h(y; \eta, b)$ satisfies the above mention equation, then, after some algebraic calculation, we can show that

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(h\left(\exp(-\eta(1-y^{-b}))-1\right)\right) = \frac{\mathrm{d}}{\mathrm{d}y}(b\eta y^{-b-1}),$$

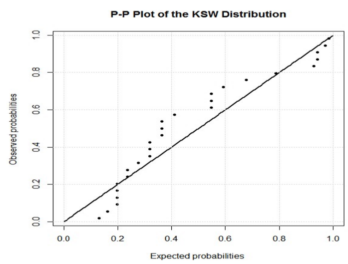
from which we arrive at

$$h(y; \eta, b) = \frac{f(y; \eta, b)}{1 - F(y; \eta, b)} = \frac{b\eta y^{-b-1}}{(\exp(-\eta(1-y^{-b}))-1)},$$

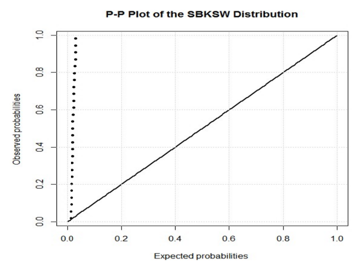
which ends the proof.□



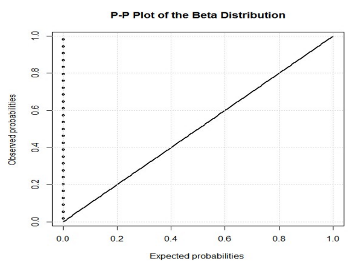
(a) UG(η, b)



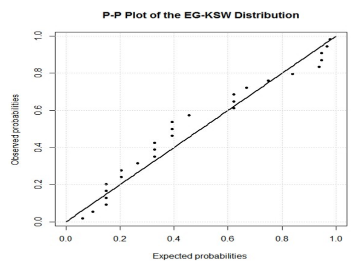
(b) KSW(α, β)



(c) SBKSW(λ, δ)



(d) Beta(α, β)



(e) EG-KSW$(\alpha, \beta, \lambda, \kappa)$

Figure 11.: PP plots for dataset 2.

Proof of Proposition 2

By definition, we get

$$\rho^r_{(m)} = \mathrm{E}(Y^r_{(m)}) = \sum_{j=n-m+1}^{n} (-1)^{j-n+m-1} \binom{j-1}{n-m} \binom{n}{j} \mathrm{I}_j\left(r\right).$$

Now, evaluate Equation (2.13). By including $F\left(y\right)$ presented in Equation (2.3), we obtain

$$\begin{aligned}\mathrm{I}_j\left(r\right) &= r \int\limits_0^1 \sum_{s=0}^{j} \binom{j}{s} \left(-1\right)^s y^{r-1} \exp\left(s\eta\left(1-y^{-b}\right)\right) \mathrm{d}y, \\ &= r \sum_{s=0}^{j} \binom{j}{s} \left(-1\right)^s \exp\left(s\eta\right) \int\limits_0^1 y^{r-1} \exp\left(s\eta y^{-b}\right) \mathrm{d}y. \qquad (1)\end{aligned}$$

Letting $t = sy^{-b}$, using the result $\mathrm{d}y = -s^{\frac{1}{b}} / \left(bt^{1+\frac{1}{b}}\right) \mathrm{d}t$ in Equation (1) and after some algebraic manipulation, we obtain

$$\begin{aligned}
\mathrm{I}_j(r) &= \frac{r}{b} \sum_{s=0}^{j} \binom{j}{s} (-1)^s s^{\frac{r}{b}} \exp(s\eta) \int_1^{\infty} t^{-\left(\frac{r}{b}+1\right)} \exp(\eta t)\, \mathrm{d}t, \\
&= \frac{r}{b} \sum_{s=0}^{j} \binom{j}{s} (-1)^s s^{\frac{r}{b}} \exp(s\eta)\, E_{\left(\frac{r}{b}+1\right)}(\eta), \\
&= \frac{r}{b} \sum_{s=0}^{j} \binom{j}{s} (-1)^s s^{\frac{r}{b}} \exp(s\eta)\, \eta^{\frac{r}{b}} \Gamma\left(-\frac{r}{b}, \eta\right).
\end{aligned}$$

By incorporating the equalities listed above together, the proof of the proposition is completed.□

Proof of Proposition 3

The LR is given by

$$\frac{f_X(y)}{f_Y(y)} = \frac{\eta_1 \exp(y^{-b}(y^b - 1)(\eta_1 - \eta_2))}{\eta_2}.$$

Thus, if $b_1 = b_2 = b$ and $\eta_1 \leq \eta_2$, then we have that

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{f_X(y)}{f_Y(y)}\right) = \frac{b \exp(y^{-b}(y^b - 1)(\eta_1 - \eta_2)) y^{-b-1} \eta_1 (\eta_1 - \eta_2)}{\eta_2} \leq 0.$$

Therefore, it indicates that $X \leq_{\mathrm{LR}} Y$, and subject to Equation (2.14), $X \leq_{\mathrm{HR}} Y$, and $X \leq_{\mathrm{ST}} Y$ are also hold.□

Appendix B: Proof of multicomponent stress-strength distribution

Next, we evaluate equation of reliability based on stress X, and strength Y, two RVs, in multi-component stress power distribution, where $Y \sim \mathrm{UG}(b, \eta_1)$ and $X \sim \mathrm{UG}(b, \eta_2)$. The process works if at the minimum s components out of k function at same time, else they crash. Equations (2.2), (2.3) and (3.16) can therefore be used to assess the reliability of the UG distribution in multicomponent stress-strength distribution as

$$\begin{aligned}
R_{s,k} &= \sum_{i=s}^{k} \binom{k}{i} \int_0^1 \left(1 - \exp\left(\eta_1\left(1 - x^{-b}\right)\right)\right)^i \left(\exp\left(\eta_1\left(1 - x^{-b}\right)\right)\right)^{k-i} \\
&\quad \times b\eta_2 x^{-b-1} \exp\left(\eta_2\left(1 - x^{-b}\right)\right) \mathrm{d}x, \\
R_{s,k} &= b\eta_2 \sum_{i=s}^{k} \sum_{j=0}^{i} (-1)^j \binom{k}{i} \binom{i}{j} \int_0^1 x^{-(b+1)} \exp\left(\left((j + k - i)\eta_1 + \eta_2\right)\left(1 - x^{-b}\right)\right) \mathrm{d}x. \quad (2)
\end{aligned}$$

By using $t = x^{-b} \Rightarrow -bx^{-b-1}\mathrm{d}x = \mathrm{d}t$ in Equation (2) and after some algebraic manipulation, we obtain

$$R_{s,k} = \eta_2 \sum_{i=s}^{k}\sum_{j=0}^{i} (-1)^j \binom{k}{i}\binom{i}{j} \exp\left(\left(j+k-i\right)\eta_1 + \eta_2\right) \int_{1}^{\infty} \exp\left(-\left(\left(j+k-i\right)\eta_1 + \eta_2\right)t\right)\mathrm{d}t.$$

Now, after solving the integral, we have

$$R_{s,k} = \sum_{i=s}^{k}\sum_{j=0}^{i} \binom{k}{i}\binom{i}{j} (-1)^j \frac{\eta_2}{\left(\left(j+k-i\right)\eta_1 + \eta_2\right)}.$$

Appendix C: Proof of the kth ordinary moment

Consider the integral given by

$$\acute{\rho}_k\left(y; b, \eta\right) = \int_0^1 y^k b\eta y^{-b-1} \exp\left(\eta\left(1 - y^{-b}\right)\right)\mathrm{d}y.$$

Allowing $t = x^{-b}$, using the result $-bx^{-b-1}\mathrm{d}x = \mathrm{d}t$, and after some algebraic manipulation, we attain at

$$\acute{\rho}_k\left(y; b, \eta\right) = \eta \exp(\eta) \int_{\infty}^{1} t^{-\frac{k}{b}} \exp(-\eta t)\left(-\mathrm{d}t\right). \tag{C1}$$

Then, after using Equation (4.18) and (4.19) in Equation (C1), we obtain given by

$$\acute{\rho}_k\left(y; b, \eta\right) = \eta \exp(\eta) E_{\frac{k}{b}}\left(\eta\right) = \eta^{\frac{k}{b}} \exp(\eta)\Gamma\left(1 - k/b, \eta\right),$$

which completes the proof.□

Author contributions

Conceptualization: A.S., J.M. and T. N. S.; data curation: G.O, J.M. and T.N.S.; formal analysis: A.S., G.O. and T.N.S.; investigation: A.S., G.O., J.M. and T.N.S.; methodology: A.S., G.O., J.M. and T.N.S.; software: B.A. and T.N.S.; validation: G.O., J.M. and T.N.S.; writing-original draft: A.S. and T.N.S.; writing-review and editing: G.O., J.M. and T.N.S. All authors have read and agreed to the published version of the article.

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Conflicts of interest

The authors declare no conflict of interest.

References

Afify, A.Z., Altun, E., Alizadeh, M., Ozel, G., and Hamedani, G.G. 2017. The odd exponentiated half-logistic-G family: properties, characterizations and applications. Chilean Journal of Statistics, 8(2), 65–91.

AL-Zaydi, A.M. 2023. Log-extended exponential-geometric distribution: Moments and inference based on generalized order statistics. Symmetry, 15, 1857.

Belili, M.C., Alshangiti, A.M., Gemeay, A.M., Zeghdoudi, H., Karakaya, K., Bakr, M.E., and Hussam, E. 2023. Two-parameter family of distributions: Properties, estimation, and applications. AIP Advances, 13, 105008.

Benkhelifa, L. 2017. The beta generalized Gompertz distribution. Applied Mathematical Modelling, 52, 341–357.

Chen, Z. 1997. Parameter estimation of the Gompertz population. Biometrical Journal, 39, 117–124.

Chesneau, C., Irshad, M.R., Shibu, D.S., Nitin, S.L., and Maya, R. 2022. On the Topp-Leone log-normal distribution: Properties, modeling, and applications in astronomical and cancer data. Chilean Journal of Statistics, 13(1), 48–67.

Chotikapanich, D., Rao, D.P., and Tang, K.K. 2007. Estimating income inequality in China using grouped data and the generalized beta distribution. Review of Income and Wealth, 53, 127–147.

Cordeiro, G.M., Mansoor, M., and Provost, S.B. 2019. The Harris extended Lindley distribution for modeling hydrological data. Chilean Journal of Statistics, 10(1), 77–94.

Cordeiro, G.M., Rodrigues, G.M., Ortega, E.M., de Santana, L.H., and Vila, R. 2022. An extended Rayleigh model: Properties, regression and COVID-19 application. arXivpreprintarXiv:2204.05214.

Das, S. and Nanda, A.K. 2013. Some stochastic orders of dynamic additive mean residual life model. Journal of Statistical Planning and Inference, 143, 400–407.

Dumonceaux, R. and Antle, C.E. 1973. Discrimination between the log-normal and the Weibull distributions. Technometrics, 15, 923–926.

Ferreira, A.A. and Cordeiro, G.M. 2023. The exponentiated power Ishita distribution: Properties, simulations, regression, and applications. Chilean Journal of Statistics, 14(1), 65–81.

Franses, P.H. 1994. Fitting a Gompertz curve. Journal of the Operational Research Society, 45, 109–113.

Gemeay, A.M., Karakaya, K., Bakr, M.E., Balogun, O.S., Atchadé, M.N., and Hussam, E. 2023. Power Lambert uniform distribution: Statistical properties, actuarial measures, regression analysis, and applications. AIP Advances, 13, 1–23.

Ghitany, M.E., Mazucheli, J., Menezes, A.F.B., and Alqallaf, F. 2019. The unit-inverse Gaussian distribution: A new alternative to two-parameter distributions on the unit interval. Communications in Statistics: Theory and Methods, 48, 3423–3438.

Gómez-Déniz, E., Sordo, M.A., and Calderín-Ojeda, E. 2014. The Log-Lindley distribution as an alternative to the beta regression model with applications in insurance. Insurance: Mathematics and Economics, 54, 49–57.

Groves-Kirkby, C.J., Denman, A.R., and Phillips, P.S. 2009. Lorenz curve and Gini coefficient: Novel tools for analysing seasonal variation of environmental radon gas. Journal of Environmental Management, 90, 2480–2487.

Gupta, R.C. 2006. Variance residual life function in reliability studies. Metron, 54, 343–345.

Jodrá, P. 2020. A bounded distribution derived from the shifted Gompertz law. Journal of King Saud University-Science, 32, 523–536.

Johnson, N. L. 1949. Systems of frequency curves generated by methods of translation. Biometrika, 36, 149–176.

Khorashadizadeh, M., Roknabadi, A.R., and Borzadaran, G.M. 2013. Doubly truncated (interval) cumulative residual and past entropy. Statistics and Probability Letters, 83, 1464–1471.

Korkmaz, M.Ç. 2020. A new heavy-tailed distribution defined on the bounded interval: the logit Slash distribution and its application. Journal of Applied Statistics, 47, 2097–2119.

Kumaraswamy, P. 1980. A generalized probability density function for double-bounded random processes. Journal of Hydrology, 46, 79–88.

Lone, S.A., Sindhu, T.N., Anwar, S., Hassan, M.K., Alsahli, S.A., and Abushal, T.A. 2023. On Construction and estimation of mixture of log-Bilal distributions. Axioms, 12(3), 309.

Makany, R. 1991. A theoretical basis for Gompertz's curve. Biometrical Journal, 33, 121–128.

Mathai, A.M., and Haubold, H.J. 2013. On a generalized entropy measure leading to the pathway model with a preliminary application to solar neutrino data. Entropy, 15, 4011–4025.

Mazucheli, J., Menezes, A.F.B., and Ghitany, M.E. 2018. The unit-Weibull distribution and associated inference. Journal of Applied Probability and Statistics, 13, 1–22.

Mazucheli, J., Menezes, A.F. and Dey, S. 2019. Unit Gompertz distribution with applications. Statistica, 79, 25–43.

Mazucheli, J., Menezes, A.F.B., and Chakraborty, S. 2019. On the one parameter unit-Lindley distribution and its associated regression model for proportion data. Journal of Applied Statistics, 46, 700–714.

Mazucheli, J., Leiva, V., Alves, B., and Menezes, A. F. 2021. A new quantile regression for modeling bounded data under a unit Birnbaum-Saunders distribution with applications in medicine and politics. Symmetry, 13, 682.

Modi, K. and Gill, V. 2020. Unit Burr-III distribution with application. Journal of Statistics and Management Systems, 23, 579–592.

Olver, F.W.J., Olde Daalhuis, A.B., Lozier, D.W., Schneider, B.I., Boisvert, R.F., Clark, C. W., Miller, B.R., and Saunders, B.V. eds. NIST Digital Library of Mathematical Functions. 2017. Online version available at http://dlmf.nist.gov/.

Pourdarvish, A., Mirmostafaee, S.M.T.K., and Naderi, K. 2015. The exponentiated Topp-Leone distribution: Properties and application. Journal of Applied Environmental and Biological Sciences, 5, 251–256.

Rao, B.R., Damaraju, C.V., Rao, P.F.R. and Damaraju, C.V. 1992. New better than used and other concepts for a class of life distributions. Biometrical Journal, 34, 919–935.

Read, C.B. 1983. Gompertz distribution. Encyclopedia of Statistical Sciences, Wiley, New York, USA.

Ribeiro-Reis, L.D., Cordeiro, G.M., and De Santana e Silva, J.J. 2022. The Mc-Donald Chen distribution: A new bimodal distribution with properties and applications. Chilean Journal of Statistics. 13(1), 68–91.

Ross, S.M., Kelly, J.J., Sullivan, R.J., Perry, W.J., Mercer, D., Davis, R.M., and Bristow, V. L. 1996. Stochastic Processes. Wiley, New York, USA.

Shafiq, A., Sindhu, T.N., Hussain, Z., Mazucheli, J., and Alves, B. 2023. A flexible probability Model for proportion data: Unit Gumbel type-II distribution, development, properties, different method of estimations and applications. Austrian Journal of Statistics, 52, 116–140.

Siddiqui, M. M. and Çağlar, M. 1994. Residual lifetime distribution and its applications. Microelectronics Reliability, 34, 211–227.

Sindhu, T.N., Shafiq, A., and Huassian, Z. 2024. Generalized exponentiated unit Gompertz distribution for modeling arthritic pain relief times data: classical approach to statistical inference. Journal of Biopharmaceutical Statistics, pages in press available at https://doi.org/10.1080/10543406.2023.2210681.

Tang, L.C., Lu, Y., and Chew, E.P. 1999. Mean residual life of lifetime distributions. IEEE Transactions on Reliability, 48, 73–78.

Willekens, F. 2001. Gompertz in context: the Gompertz and related distributions. In Forecasting mortality in developed countries: insights from a statistical, demographic and epidemiological perspective. Springer, Dordrecht, 105–126.

Wu, J.W. and Lee, W.C. 1999. Characterization of the mixtures of Gompertz distributions by conditional expectation of order statistics. Biometrical Journal: Journal of Mathematical Methods in Biosciences, 41, 371–381.

Yildirim, E., Ilikkan, E.S., Gemeay, A.M., Makumi, N., Bakr, M.E., and Balogun, O.S. 2023. Power unit Burr-XII distribution: Statistical inference with applications. AIP Advances, 13, 105107.