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Contingency Tables
Research Paper

Orthogonal decomposition of symmetry model using sum-symmetry model for ordinal square contingency tables

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Abstract

This study focuses on R × R ordinal square contingency tables. Ordinal square contingency tables are always obtained by cross-classifying the matched-pair data of the two ordinal categorical variables with the same classifications. A well-known model in square contingency tables is the symmetry model. This study focuses on the relationship between the symmetry and sum-symmetry models. The sum-symmetry model has a symmetric structure between the probability that the sum of row variable X and column variable Y is t, when $X < Y$, for $t = 3, \dots, 2\mathrm{R} - 1$ and the probability that the sum of X and Y is t, when $X > Y$. The sum-symmetry model inevitably holds when the symmetry model holds, but the converse is not necessarily true. This study proposes a model that must be satisfied in addition to the sum-symmetry model, to satisfy the symmetry model. We also reveal that the value of the likelihood ratio chi-squared statistic of the symmetry model is equal to the sum of chi-squared statistic of the sum- and sum-parameter symmetry models. We evaluate the utility of these properties by applying them to real-world vision data.

Keywords: Matched-pair data · Necessary and sufficient condition · Ordinal categorical data · Sum-parameter symmetry · Test statistic.

Mathematics Subject Classification: Primary 62H17 · Secondary 62F03.

1. Introduction

Contingency tables are widely used in many disciplines, including data science, engineering and scientific research, see, Agresti (2013), Vélez and Marmolejo-Ramos (2017).

This study focuses on R × R ordinal square contingency tables. Ordinal square contingency tables are always obtained by cross-classifying the matched-pair data of the two ordinal categorical variables with the same classifications. For such data, we examine whether the probability of the observations falling in the (i, j)th cell of the table, when $i < j$, is equal to the probability of the observations falling in the (j, i)th cell. In other words, we analyze whether there is symmetry in cell probability in regard to the main diagonal cells of the table. The symmetry (S) model proposed by Bowker (1948) is useful for analyzing

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the above structure. Other models, with weaker constraints than the S model, have also been proposed, for example, McCullagh (1978) introduced the conditional symmetry (CS) model, while Read (1977) introduced the global symmetry (GS) model. Moreover, Read (1977) revealed (i) the decomposition theorem—the S model holds, if and only if, both the CS and GS models hold—, and (ii) the orthogonality of the test statistics—the value of the likelihood ratio chi-squared statistic (denoted by G^2) of the S model is equal to the sum of G^2 of the CS and GS models. This decomposition theorem is useful to evaluate the cause that the symmetry model does not hold—decomposition theorem is one of the priority disciplines of the research on square contingency tables.

Yamamoto et al. (2013) introduced the sum-symmetry (SS) and conditional sum-symmetry (CSS) models. Yamamoto et al. (2013) also revealed that (i) the decomposition theorem—the SS model holds, if and only if, both the CSS and GS models hold—, and (ii) the orthogonality of the test statistics for this decomposition theorem.

This study focuses on the relationship between the S and SS models. The SS model inevitably holds when the S model holds, but the converse is not necessarily true. We want to propose a model that must be satisfied in addition to the SS model, to satisfy the S model. Moreover, we reveal that the value of G^2 of the S model is equal to the sum of G^2 of the SS and proposed models.

The rest of this paper is organized as follows. Section 2 introduces the S, SS, and proposed models, and gives the orthogonality of the test statistics—the value of G^2 of the S model is equal to the sum of G^2 of the SS and proposed models. Section 3 evaluates the utility of the properties given by this study by applying them to real-world data. Section 4 closes with concluding remarks.

2. Orthogonal decomposition of the statistical model

2.1 Statistical model

In this section, first, we describe the models introduced in Section 1.

Let X and Y be the row and column variables, respectively. The S model is defined by

$$\mathrm{P}(X = i, Y = j) = \mathrm{P}(X = j, Y = i) \quad (i < j).$$

The S model indicates that the probability of the observations falling in the (i, j)th cell of the table, when $i < j$, is equal to the probability of the observations falling in the (j, i)th cell. Thus, the S model indicates the symmetric structure of the cell probabilities in regard to the main diagonal cells of the table.

The CS model is defined by

$$\mathrm{P}(X = i, Y = j) = \Delta \mathrm{P}(X = j, Y = i) \quad (i < j).$$

The CS model indicates that the probability of the observations falling in the (i, j)th cell of the table, when $i < j$, is Δ times higher than the probability of the observations falling in the (j, i)th cell. Thus, the CS model indicates the asymmetric structure of the cell probabilities in regard to the main diagonal cells of the table. Note that the CS model with $\Delta = 1$ is equivalent to the S model. Therefore, the CS model inevitably holds when the S model holds, but the converse is not necessarily true.

Read (1977) showed that the GS model must be satisfied in addition to the CS model, to satisfy the S model. The GS model is defined by

$$\mathrm{P}(X < Y) = \mathrm{P}(X > Y).$$

Yamamoto et al. (2013) proposed the SS and CSS models. The SS model is defined by

$$\mathrm{P}(X+Y=t, X<Y) = \mathrm{P}(X+Y=t, X>Y) \quad (t=3,\ldots,2\mathrm{R}-1).$$

The SS model indicates that the probability that the sum of X and Y is t when $X<Y$, for $t=3,\ldots,2\mathrm{R}-1$ is equal to the probability that the sum of X and Y is t when $X>Y$.

The CSS model is defined by

$$\mathrm{P}(X+Y=t, X<Y) = \Delta\mathrm{P}(X+Y=t, X>Y) \quad (t=3,\ldots,2\mathrm{R}-1).$$

The CSS model indicates that the probability that the sum of X and Y is t when $X<Y$, for $t=3,\ldots,2\mathrm{R}-1$ is Δ times higher than the probability that the sum of X and Y is t when $X>Y$. Note that the CSS model with $\Delta=1$ is equivalent to the SS model. Therefore, the CSS model inevitably holds when the SS model holds. However, the converse is not necessarily true. Yamamoto et al. (2013) also showed that the GS model must be satisfied, in addition to the CSS model, to satisfy the SS model.

Although the details are omitted, in recent years, there has been some researches on models with respect to the sum of row and column variables. For example, see Yamamoto et al. (2016), Iki and Tomizawa (2020), Ando (2021a,b).

This study focuses on the relationship between the S and SS models. When the number of categories R is less than or equal to three, the SS model is equivalent to the S model. When $\mathrm{R}=3$, the SS model is expressed as follows:

$$\begin{aligned}
\mathrm{P}(X+Y=3, X<Y) &= \mathrm{P}(X+Y=3, X>Y)\\
&\Leftrightarrow \mathrm{P}(X=1, Y=2) = \mathrm{P}(X=2, Y=1),\\
\mathrm{P}(X+Y=4, X<Y) &= \mathrm{P}(X+Y=4, X>Y)\\
&\Leftrightarrow \mathrm{P}(X=1, Y=3) = \mathrm{P}(X=3, Y=1) \quad \text{and}\\
\mathrm{P}(X+Y=5, X<Y) &= \mathrm{P}(X+Y=5, X>Y)\\
&\Leftrightarrow \mathrm{P}(X=2, Y=3) = \mathrm{P}(X=3, Y=2).
\end{aligned}$$

However, when $R=4$, the SS model is expressed as follows:

$$\begin{aligned}
\mathrm{P}(X+Y=3, X<Y) &= \mathrm{P}(X+Y=3, X>Y)\\
&\Leftrightarrow \mathrm{P}(X=1, Y=2) = \mathrm{P}(X=2, Y=1),\\
\mathrm{P}(X+Y=4, X<Y) &= \mathrm{P}(X+Y=4, X>Y)\\
&\Leftrightarrow \mathrm{P}(X=1, Y=3) = \mathrm{P}(X=3, Y=1),\\
\mathrm{P}(X+Y=5, X<Y) &= \mathrm{P}(X+Y=5, X>Y)\\
&\Leftrightarrow \mathrm{P}(X=1, Y=4) + \mathrm{P}(X=2, Y=3)\\
&\quad\ = \mathrm{P}(X=4, Y=1) + \mathrm{P}(X=3, Y=2),\\
\mathrm{P}(X+Y=6, X<Y) &= \mathrm{P}(X+Y=6, X>Y)\\
&\Leftrightarrow \mathrm{P}(X=2, Y=4) = \mathrm{P}(X=4, Y=2) \quad \text{and}\\
\mathrm{P}(X+Y=7, X<Y) &= \mathrm{P}(X+Y=7, X>Y)\\
&\Leftrightarrow \mathrm{P}(X=3, Y=4) = \mathrm{P}(X=4, Y=3).
\end{aligned}$$

We see that when R is more than or equal to four, the SS model is different from the S model.

Therefore, when R is more than or equal to four, we want to propose a model that must be satisfied, in addition to the SS model, to satisfy the S model. Moreover, as in Read (1977) and Yamamoto et al. (2013), the proposed model should satisfy the assumption that the value of G^2 of the S model is equal to the sum of G^2 of the SS and proposed models.

We propose the sums-parameter symmetry (SPS) model as the model that must be satisfied, in addition to the SS model, to satisfy the S model. The SPS model is defined by

$$\mathrm{P}(X = i, Y = j) = \Delta_{i+j}\mathrm{P}(X = j, Y = i) \quad (i < j). \tag{2.1}$$

The SPS model indicates that the probability of the observations falling in the (i, j)th cell of the table, when $i < j$, is Δ_{i+j} times higher than the probability of the observations falling in the (j, i)th cell. The SPS models with $\{\Delta_{i+j} = 1\}$ and $\{\Delta_{i+j} = \Delta\}$ are equivalent to the S and CS models, respectively. When we replace $\{\Delta_{i+j}\}$ with $\{\Delta_{j-i}\}$ in Equation (2.1), the model is equivalent to the diagonals-parameter symmetry model given by Goodman (1979). Moreover, when we replace $\{\Delta_{i+j}\}$ with $\{\Delta^{j-i}\}$ in Equation (2.1), the model is equivalent to the linear diagonals-parameter symmetry model given by Agresti (1983).

2.2 Orthogonal decomposition

In this section, we present the orthogonal decomposition of the S model using SS and SPS models. We obtain the following decomposition theorem.

Theorem 2.1 The following necessary and sufficient condition holds:

The S model holds, if and only if, both the SS and SPS models hold.

Proof It is clear that the necessary condition (i.e., if the S model holds, then both the SS and SPS models hold) holds. We need to show that the sufficient condition (i.e., if both SS and SPS models hold, then the S model holds) also holds. From the SPS model holds, the following equality also holds:

$$\sum_{(i,j)\in A_t}\sum \mathrm{P}(X = i, Y = j) = \Delta_t \sum_{(i,j)\in A_t}\sum \mathrm{P}(X = j, Y = i) \quad (t = 3, \dots, 2\mathrm{R} - 1), \tag{2.2}$$

where $A_t = \{(i, j)|i + j = t, i < j\}$.

The SS model can be also expressed as follows:

$$\sum_{(i,j)\in A_t}\sum \mathrm{P}(X = i, Y = j) = \sum_{(i,j)\in A_t}\sum \mathrm{P}(X = j, Y = i) \quad (t = 3, \dots, 2\mathrm{R} - 1). \tag{2.3}$$

From Equation (2.2) and (2.3), we obtain $\Delta_t = 1$ for all $t = 3, \dots, 2\mathrm{R} - 1$. As the SPS model with $\{\Delta_t = 1\}$ is equivalent to the S model, the sufficient condition holds. The proof is complete. ■

Theorem 2.1 is useful for demonstrating the cause that the S model does not hold for the presented data.

We obtain the following theorem from Theorem 2.1 and the decomposition theorem of Yamamoto et al. (2013) (i.e., the SS model holds if and only if both CSS and GS models hold).

Theorem 2.2 The following necessary and sufficient condition holds:

The S model holds if and only if all CSS, GS, and SPS models hold.

We denote n_{ij} as the observed frequency in the (i, j)th cell of the table $(i = 1, \ldots, \mathrm{R}; j = 1, \ldots, \mathrm{R})$. We assume multinomial sampling over the cells of the table.

Each model can be tested for goodness-of-fit by, for example, the test statistic G^2 with the corresponding degrees of freedom. The G^2 of model M is given by

$$\mathrm{G}^2(\mathrm{M}) = 2\sum_{i=1}^{R}\sum_{j=1}^{R} n_{ij} \log\left(\frac{n_{ij}}{\hat{e}_{ij}}\right),$$

where $\hat{e}_{ij}$ is the maximum likelihood estimate (MLE) of the expected frequency e_{ij} under model M. The number of degrees of freedom for the S, SS, and SPS models are $\mathrm{R}(\mathrm{R}-1)/2$, $2\mathrm{R}-3$, and $(\mathrm{R}-2)(\mathrm{R}-3)/2$, respectively. It must be noted that the number of degrees of freedom for the S model is equal to the sum of the number of degrees of freedom for the SS and SPS models.

We obtain the following orthogonality of test statistic.

Theorem 2.3 The following equality holds:

$$\mathrm{G}^2(\mathrm{S}) = \mathrm{G}^2(\mathrm{SS}) + \mathrm{G}^2(\mathrm{SPS}).$$

Proof Although the details are omitted, the $\hat{e}_{ij}$ under the S, SS, and SPS models are provided in Equation (2.4), (2.5), and (2.6), respectively.

$$\hat{e}_{ij} = \begin{cases} \dfrac{n_{ij} + n_{ji}}{2} & (i \neq j), \\ n_{ij} & (i = j), \end{cases} \tag{2.4}$$

$$\hat{e}_{ij} = \begin{cases} \dfrac{(B_t + C_t)}{2B_t} n_{ij} & (i + j = t, i < j), \\ n_{ij} & (i \neq j), \\ \dfrac{B_t + C_t}{2C_t} n_{ij} & (i + j = t, i > j), \end{cases} \tag{2.5}$$

where

$$B_t = \sum\sum_{(i,j)\in A_t} n_{ij} \quad \text{and} \quad C_t = \sum\sum_{(i,j)\in A_t} n_{ji}.$$

$$\hat{e}_{ij} = \begin{cases} \dfrac{B_t}{B_t + C_t}(n_{ij} + n_{ji}) & (i + j = t, i < j), \\ n_{ij} & (i \neq j), \\ \dfrac{C_t}{B_t + C_t}(n_{ij} + n_{ji}) & (i + j = t, i > j). \end{cases} \tag{2.6}$$

$n_{ij}/\hat{e}_{ij}$ in the S model is equal to the product of that of the SS and SPS models. Therefore, the value of $G^2(S)$ is equal to the sum of $G^2(SS)$ and $G^2(SPS)$. The proof is complete. ■

From Theorem 2.3, we point out that the value of $G^2(S)$ assuming that the SS model holds true (i.e., $G^2(S|SS)$) is equal to the value of $G^2(SPS)$ because $G^2(S|SS) = G^2(S) - G^2(SS)$.

We obtain the following theorem from Theorem 2.3 and the orthogonality of the test statistic of Yamamoto et al. (2013) (i.e., $G^2(SS) = G^2(CSS) + G^2(GS)$).

Theorem 2.4 The following necessary and sufficient condition holds:

$$G^2(S) = G^2(CSS) + G^2(GS) + G^2(SPS).$$

3. Application to real-world data

First, we consider the data set in Table 1, taken from Tomizawa (1985). This data set presents a cross-classification of vision grades for right and left eyes. Table 1 is the data of unaided distance vision of 4746 university students aged 18 to 25, including about 10% of the women of the Faculty of Science and Technology, Tokyo University of Science, examined in 1982.

The X is the right eye grade, and Y is the left eye grade with the categories ordered from the highest (1) to the lowest grade (4). Yamamoto et al. (2013) mentioned that it is natural to evaluate the degree of an individual's vision grade as the sum of the grades of both right and left eyes for these data. The sum of the grades of both the right and left eyes (i.e., $X + Y$) is ordered from the highest (2) to the lowest grade (10). Table 2 gives the values

Table 1. The table below is the data of unaided distance vision of 4746 university students aged 18–25, including 10% of the women of the Faculty of Science and Technology, Tokyo University of Science, examined in 1982; source Tomizawa (1985).

Right eye grade	Left eye grade (1)	(2)	(3)	(4)	Total
Highest (1)	1291	130	40	22	1483
	(1291)	(130)	(40)	(20.40)	
Second-highest (2)	149	221	114	23	507
	(149)	(221)	(115.60)	(23.0)	
Third-highest (3)	64	124	660	185	1033
	(64)	(122.40)	(660)	(185)	
Lowest (4)	20	25	249	1429	1723
	(21.60)	(25)	(249)	(1429)	
Total	1524	500	1063	1659	4746

Note: Estimates under the SPS model are shown in parentheses in the second line.

of G^2 for the S, SS, and SPS models. This table shows that the SPS model fits well, but the other models fit poorly. Table 1 shows the MLEs of the expected frequencies under the SPS model. From Theorem 2.1, we can infer that the S model does not hold for the data in Table 1 due to the SS model, rather than the SPS model. Under the SPS model, the MLE of $\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7$ are 0.872, 0.625, 0.944, 0.920, 0.743, respectively. The S model does not hold, largely due to the asymmetric structure corresponding to the cells in which the sums of the right and left eye grades are 4 and 7.

Next, we consider the data in Table 3, taken from Stuart (1953). Table 3 is the data of unaided distance vision of 7477 women aged 30–39, employed in Royal Ordnance factories from 1943 to 1946.

Table 2. Values of the likelihood ratio chi-squared statistic G^2 for each model applied to the data in Table 1

Applied models	Degrees of freedom	G^2
S	6	16.955*
SS	5	16.668*
SPS	1	0.287

$*$ indicates significance at 0.05 level.

Table 4 gives the values of G^2 for the S, SS, CSS, GS, and SPS models. This table shows that the CSS model fits well, but the others fit poorly. The parenthesized values in Table 3 are the MLEs of the expected frequencies under the CSS model.

Table 3. The table below is the data of unaided distance vision of 7477 women aged 30–39 employed in Royal Ordnance factories from 1943 to 1946; source Stuart (1953).

	Left eye grade				
Right eye grade	(1)	(2)	(3)	(4)	Total
Highest (1)	1520	266	124	66	1976
	(1520)	(268.45)	(129.40)	(63.76)	
Second-highest (2)	234	1512	432	78	2256
	(231.55)	(1512)	(417.31)	(85.91)	
Third-highest (3)	117	362	1772	205	2456
	(111.60)	(377.40)	(1772)	(206.17)	
Lowest (4)	36	82	179	492	789
	(37.53)	(74.09)	(117.83)	(492)	
Total	1907	2222	2507	841	7477

Note: Estimates under the conditional sum-symmetry (CSS) model are shown in the parentheses in the second line.

Table 4. Values of the likelihood ratio chi-squared statistic G^2 for each model applied to the data in Table 3

Applied models	Degrees of freedom	G^2
S	6	19.249*
SS	5	15.299*
CSS	4	3.403
GS	1	11.896*
SPS	1	3.951*

* indicates significance at 0.05 level.

From Theorem 2.1, we can infer that the S model does not hold for the data in Table 3 due to both the SS and SPS models. We are also interested in finding why the SS model does not hold. From the decomposition theorem of Yamamoto et al. (2013), we can infer that the SS model does not hold because of the GS model, rather than the CSS model. Moreover, from Theorem 2.2, we can infer that the S model does not hold because of the GS and SPS models, rather than the CSS model.

Under the CSS model, since the MLE of Δ is 1.159, the probability that the degree of the individual's vision grade, in which the right eye grade is greater than the left eye grade is t for $t = 3, 4, 5, 6, 7$, is estimated to be 1.159 times higher than the probability that the degree of the individual's vision grade in which the right eye grade is greater than the left eye grade is t. Therefore, under the CSS model, a woman's right eye vision was estimated to be better than her left eye vision.

4. Concluding remarks

When R is less than or equal to three, the sum-symmetry model is equivalent to the symmetry model. On the other hand, when R is more than or equal to four, the sum-symmetry model is different from the symmetry model. Therefore, when R is more than or equal to four, we are interested in finding a model which must be satisfied in addition to the sum-symmetry model, to satisfy the symmetry model.

This study reveals that, (i) it is necessary to satisfy the sums-parameter symmetry model in addition to the sum-symmetry model, to satisfy the symmetry model (i.e., Theorem 2.1) and (ii) the value of G^2 of the symmetry model is equal to the sum of G^2 of the sum-symmetry and sums-parameter symmetry models (i.e., Theorem 2.3). Theorem 2.1 is useful for evaluating why the symmetry model does not hold for the presented data, as shown in Section 3. From Theorem 2.3, we show that the value of $\mathrm{G}^2(\mathrm{S})$, assuming the sum-symmetry model holds true (i.e., $\mathrm{G}^2(\mathrm{S}|\mathrm{SS})$) is equal to the value of $\mathrm{G}^2(\mathrm{SPS})$, because $\mathrm{G}^2(\mathrm{S}|\mathrm{SS}) = \mathrm{G}^2(\mathrm{S}) - \mathrm{G}^2(\mathrm{SS})$.

Generally, we assume that, (i) model M_1 holds if and only if models M_2 and M_3 both hold, and (ii) the following asymptotic equivalence holds:

$$G^2(\mathrm{M}_1) \simeq G^2(\mathrm{M}_2) + G^2(\mathrm{M}_3), \tag{4.7}$$

where the number of degrees of freedom for model M_1 is equal to the sum of the number of degrees of freedom for models M_2 and M_3. Darroch and Silvey (1963) described that, (i) when Equation (4.7) holds, if both models M_2 and M_3 are accepted (at the α significance level) with high probability, then model M_1 would be accepted; but (ii) when that does not hold, such an incompatible situation where both models M_2 and M_2 are accepted with high probability, but model M_1 is rejected with high probability, is quite possible. In fact, Darroch and Silvey (1963) and Tahata et al. (2011) showed such interesting examples. We note that the proposed decomposition theorems satisfy Equation (4.7).

Although the proposed decomposition theorems were applied to vision data in real-world data analysis, we believe that they are also useful for analyzing other data, for example, social mobility data.

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Appendix. Example R code

We provide an example R code to obtain the maximum likelihood estimate of the expected frequency under the S, SS, and SPS models. The R code includes the following functions: S_MLE(x), SS_MLE(x), and SPS_MLE(x), where x is an $R \times R$ ordinal square contingency table.

```
S_MLE <- function(x){
```

```
  R <- nrow(x)
  e <- matrix(0,nrow=R, ncol=R)
  for(i in 1:R){
    for(j in 1:R){
      if(i!=j){
        e[i,j] <- (x[i,j]+x[j,i])/2
      }
      else{
        e[i,j] <- x[i,j]
      }
    }
  }
  return(e)
}

SS_MLE <- function(x){
          R <- nrow(x)
          e <- matrix(0,nrow=R, ncol=R)
          A <- rep(0,2*R-1)
          B <- rep(0,2*R-1)
          for(i in 1:R){
            for(j in 1:R){
              if(i < j){
                A[i+j] <- A[i+j] + x[i,j]
              }
              else if(i > j){
                B[i+j] <- B[i+j] + x[i,j]
              }
            }
          }
          for(i in 1:R){
                    for(j in 1:R){
                      if(i < j){
                        e[i,j] <- x[i,j]*(A[i+j]+B[i+j])/(2*A[i+j])
                      }
                      else if(i==j){
                        e[i,j] <- x[i,j]
                      }
                      else if(i > j){
                        e[i,j] <- x[i,j]*(A[i+j]+B[i+j])/(2*B[i+j])
                      }
                    }
          }
          return(e)
}

SPS_MLE <- function(x){
  R <- nrow(x)
  e <- matrix(0,nrow=R, ncol=R)
  A <- rep(0,2*R-1)
  B <- rep(0,2*R-1)
```

```
  for(i in 1:R){
    for(j in 1:R){
      if(i < j){
        A[i+j] <- A[i+j] + x[i,j]
      }
      else if(i > j){
        B[i+j] <- B[i+j] + x[i,j]
      }
    }
  }
  for(i in 1:R){
    for(j in 1:R){
      if(i < j){
        e[i,j] <- (x[i,j]+x[j,i])*A[i+j]/(A[i+j]+B[i+j])
      }
      else if(i==j){
        e[i,j] <- x[i,j]
      }
      else if(i > j){
        e[i,j] <- (x[i,j]+x[j,i])*B[i+j]/(A[i+j]+B[i+j])
      }
    }
  }
  return(e)
}
```