Chilean Journal of Statistics
Volume 13, Number 2 (December 2022), Pages 201–220
DOI: 10.32372/chjs.13-02-05

Distribution Theory
Research Paper

POT-based estimator of the ruin probability in infinite time for loss models: An application to insurance risk

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(Received: 24 June 2021 · Accepted in final form: 17 May 2022)

Abstract

Asymptotic Ruin Probabilities in the Cramér-Lundberg model has been widely studied when the claims have light-tailed or heavy-tailed distributions. However, it has not been studied for extreme values over some threshold. In this paper, we investigate the use of the peaks over threshold method, to construct a new estimator of the ruin probability for a risk process with heavy tails claims amounts with infinite variance for the stationary arrival claims in an infinite time. Our approach is based on approximating the sample over some threshold by the generalized Pareto distribution. We prove that the proposed estimator is consistent and asymptotically normal. The performance of our new estimator is illustrated by some results of simulations for some loss models and provides an extensive example application to Danish data on large fire insurance.

Keywords: Risk Process · Extremes values · Heavy-tailed distribution · Generalised Pareto Distribution (GPD) · Ruin probability · POT method.

Mathematics Subject Classification: 62E20 · 62F12 · 62G32 · 62P05.

1. Introduction

Risk theory in general and ruin probabilities in particular are traditionally considered as part of insurance mathematics. The ruin probability refers to the risk that the monetary surplus of an insurance company becomes less than zero. The pioneering ideas of Filip Lundberg's Lundberg (1903) thesis remain the basis for the collective risk theory for general insurance actuaries and the estimation of the ruin probability. Lundberg's work was republished in the 1930s by Harald Cramer Cramer (1930).

In recent years, the classical risk process has been extended to more practical and real situations. The Cramér-Lundberg model has been generalized in two main ways. i) Assumptions on the company's liabilities (on the modeling of claims): heavy/light tails, dependency between claims etc., dependency between claims number process and claims, regime change, etc, and, ii) Asset assumptions instead of the fixed premium rate, various strategies (divi-

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ISSN: 0718-7912 (print)/ISSN 0718-7920 (online)
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http://www.soche.cl/chjs

dend payment, reserve investment, interest rate consideration) instantaneous, Brownian or Levy type perturbations on prices, etc.

In classical risk model, the claim number process was assumed to be a Poisson process and the individual claim amounts were described as independent and identically distributed random variables. The claim modeling is an important procedure that leads to the pricing of premium and risk analysis for an insurance company.

Recent studies concluded that simple analytic results for the ruin probability using the classical model exist when the claim amounts distribution is exponential or close to it, but for other claims amounts distributions, they are not easy to obtain.

Modeling insurance loss data of a unimodal type with a heavy tail has been an interesting topic for actuaries. Distributions that can mimic the heavy tail of the insurance loss data are crucial to sufficiently provide a good estimate of the associated business risk level (Abubakar et al., 2015). In insurance, heavy tails are encountered when modeling for instance fire and storm damages (Embrechts et al, 1997).

A substantial bibliography is available on quantile estimation of heavy-tailed distributions, see Embrechts et al (1997) for an excellent introduction and overview of this field.

The literature on the ruin theory is rich in methods for calculating, see, e.g., Sundt et al (1995), approximating, see, e.g., De Vylder (1978), asymptotically analyzing the ruin probabilities, see, e.g., Aurzada et al. (2020). Some are in the univariate, see, e.g., Asmussen (2010), Mikosch (2000), and even multivariate cases, see, e.g., Behme et al (2020). Others are in the Stationary and even Non-Stationary cases, see, e.g., Xiaoping et al. (2014), Zhu (2013). We also find several publications using the Bayesian approach, see, e.g., Concepcion et al. (2007).

In addition, ruin theory has fruitful methodological links and applications to other fields of applied probability, like queueing theory and mathematical finance (pricing of barrier options, credit products, etc.).

The common statistical techniques used in the modeling of insurance claims are related to average. However, some of these claims should be considered extreme rather than average. Therefore, the extreme value theory is of particular importance to insurance mathematics, it studies extreme events that, although low in frequency, cause high claims to the insurance companies. There are two ways to model extreme events using extreme value models. The first approach is to divide the sample into blocks and then obtain the maximum of each block, which is called the block maxima method. The second approach is the Peaks Over the Threshold method (POT), which involves taking the large observations that exceed some threshold u. POT models are generally useful for practical applications, see, for example, Reiss et al. (2002).

In this paper, we are interested in the estimator of the ruin probability for the risk process with claims of heavy tails. Our approach is based on an extreme theory using the Peaks Over the Threshold method (POT) for the approximate excess by generalized Pareto distribution (GPD). We attempt to estimate the ruin probability in infinite time with a large initial reserve. We propose a new estimator of the ruin probability when the second moment of the claims is infinite and we provide the almost sure consistency and the asymptotic normality of our estimator.

The paper is organized as follows. In Section 2, we present the classical risk model and we give the definition of the ruin probability when the claims are heavy-tailed, and we present some illustration for the performance and the normality of the ruin probability when the variance of the claims are finite. Section 3 is restricted to the Generalized Pareto Distribution using the Peaks Over Threshold method. In section 4, we give some assumptions and state an almost sure consistency and asymptotic normality for the proposed estimator of the ruin probability when the second moment of the claims is finite or infinite. Section 5 is devoted to validating the performance of our results by some simulation results and provides an example of extended application to Danish large fire insurance loss data. The appendix is reserved for the proofs of the results. All computations and graphics presented in this paper were done in the `R` software.

2. Model

2.1 Background

Let the initial capital of the insurance company is denoted by u. The number of claims in the time period $(0, t]$, denoted by $(N_t)_{t\geq 0}$, is described by a Poisson process with fixed rate $\lambda > 0$. Claim severities are non-negative random variables (RV's), given by an independent and identically distributed (i.i.d.) sequence $X, X_1, X_2, ...,$ having distribution function (CDF) F with unknown mean $\mu_X < +\infty$ and variance σ_X^2. We assume that $X, X_1, X_2, ...,$ are independent of $(N_t)_{t\geq 0}$. We also assume that the insurance company receives a premium at a constant rate p per unit time and that the so-called net-profit condition holds, that is $\rho = p/\lambda > \mu_X$, (see, Asmussen (2010)). The classical risk process $(R_t)_{t\geq 0}$ is given by

$$R_t = u + pt - \sum_{i=1}^{N_t} X_i, \quad t \geq 0.$$

The corresponding claim surplus process is defined by

$$S_t = u - R_t = \sum_{i=1}^{N_t} X_i - pt, \quad t \geq 0.$$

We are primarily interested in the probability that S_t exceeds an initial reserve u at some time t prior to or at a horizon time T. Explicitly, this probability may be written as

$$\Phi(u, T) = P[\sup_{0<t\leq T} \mathrm{S}_t > u].$$

The ruin probability in infinite time is defined by

$$\Phi(u) = \lim_{T\to+\infty} \Phi(u, T).$$

We are interested in the case when $X, X_1, X_2, ...,$ have heavy tails.

Let X be a positive random variable with distribution function F, where

$$\bar{F}(x) = cx^{-\frac{1}{\xi}}(1 + x^{-\delta}\mathrm{L}(x)), \quad as \quad x \uparrow +\infty, \tag{2.1}$$

and $\bar{F}(x) = 1 - F(x)$ represent the tail function of the distribution F.

For $\xi \in (0, 1)$, $\delta > 0$, and some real constant c, with L a slowly varing function at infinity, that $\mathrm{L}(tx)/\mathrm{L}(x) \to 1$, as $x \uparrow +\infty$ for $t > 0$. For more details on these function, see, e.g., Chapter 0 in Resnik (1987) or Seneta (1976).

F is regularly varying tail at $+\infty$ with index ξ, that $\bar{\mathrm{F}}(tx)/\bar{\mathrm{F}}(x) \to t^{(-1/\xi)}$, as $x \uparrow +\infty$ uniformly for $t > 0$.

F is called heavy tail with tail index $\xi > 0$ (e.g., Pareto, Burr, log-Normal, log-Gamma, Student, etc..), see, e.g., (Dekkers et al, 1993, 1989). Notice that when $\xi \in (0, 1/2)$, we have $\mu_X = E(X) < \infty$ and $E(X^2) < \infty$. But if $\xi \in (1/2, 1)$, we have $E(X^2) = \infty$.

Any F with a regularly varying tail is subexponential, i.e. $\bar{F}^{*2}(x)/\bar{F}(x) \to 2$, as $x \uparrow +\infty$.

Here $F * G$ denotes the convolution of the distribution functions F and $G, F^{*1} = F$ and $F^{*(n+1)} = F^{*n} * F$, (see Asmussen (2010))

If F is subexponential, it has been shown that for large initial reserve u, the ruin probability $\Phi(u)$, (see, e.g., K Klüppelberg et al. (1996), Asmussen (2010), Vladimir et al (2000) and Zhu (2013)), given by

$$\lim_{u \longrightarrow +\infty} \frac{\Phi(u)}{\bar{F}_0(u)} = \frac{\mu_X}{(\rho - \mu_X)},$$

where F_0 denotes the stationary excess distribution or the integrated tail distribution,

$$F_0(x) = \frac{1}{\mu_X} \int_0^x \bar{F}(x) dx, \quad x \geq 0.$$

We can then write the approximation of the ruin probability $\Phi(u)$ as follows.

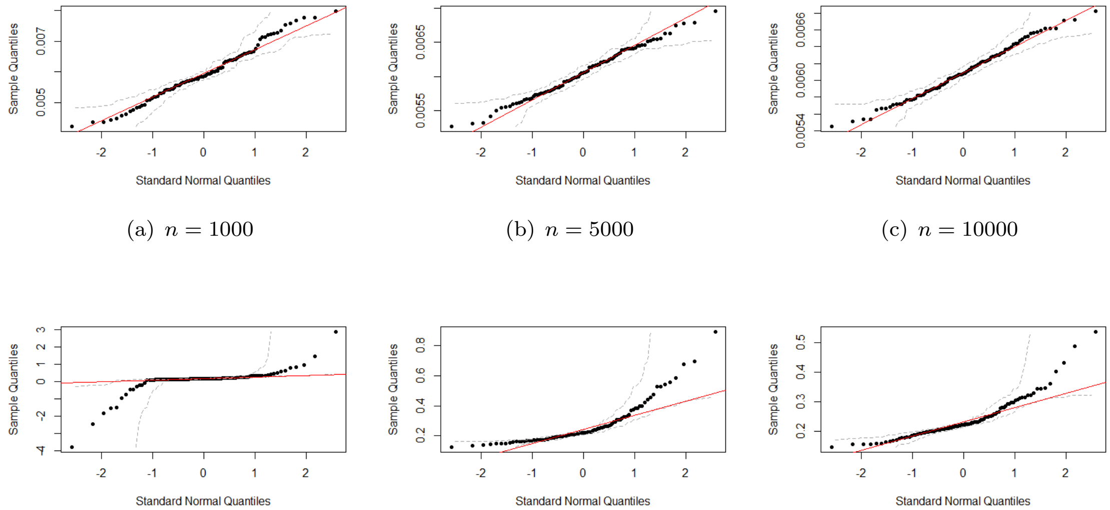
$$\Phi(u) \simeq \frac{1}{(\rho - \mu_X)} \int_u^{+\infty} \bar{F}(x) dx \text{ for a large initial reserve } u. \tag{2.2}$$

2.2 Traditional Estimator of the Ruin Probability

Let $X_1, ..., X_n$ be a random sample of size n of CDF F. The Tail of the distribution F assumed to start at some level u_n supposed sufficiently high, and we set $Y_i = max(X_i - u_n, 0)$ with CDF F_Y. By replacing F and μ_X by its empirical estimators $\hat{F}_n$ and $\hat{\mu}_X$ respectively in Equation (2.2), the estimator is written as follows:

$$\hat{\Phi}_n(u_n) = \frac{1}{(\rho - \hat{\mu}_X)} \int_{u_n}^{+\infty} \hat{\bar{F}}_n(x) dx \text{ for a large initial reserve } u_n. \tag{2.3}$$

In Section 4, we prove that the estimator given in Equation (2.3) is asymptotically normal provided that it has a finite second moment. To illustrate the performance and normality of $\hat{\Phi}_n(u_n)$, we draw samples from the Pareto distribution $\bar{F}(x) = x^{-1/\xi}$, $x > 1$, $\xi > 0$ using `R` packages (`actuar`, `evir`, `extremefit`, `tea`, `boot`) to generate samples, and to estimate parameters of the GPD distribution. The `qqplot` and `boostrap` functions were used to adjust graphically the normality of the ruin probability. We choose two values $\xi = 1/4$ in which case we have the two moments are finite and $\xi = 3/4$ in which case we have the second moment is infinite as shown in Figure 1. We can notice that the ruin probability is well fitted by the normal distribution for $\xi = 1/4$ (finite variance). On the other hand, if $\xi = 3/4$ (infinite variance), the ruin probability is far from the normal distribution (see Figure 1).



(a) $n = 1000$ (b) $n = 5000$ (c) $n = 10000$

(d) $n = 1000$ (e) $n = 5000$ (f) $n = 10000$

Figure 1. QQ-plot of the Ruin probability against the normal distribution when $\xi = 1/4$ (top row) and $\xi = 3/4$ (bottom row)

3. The GPD Estimate

3.1 GPD Approximation of the Tail of Distribution

Let F_Y be the distribution of the excesses over the threshold u_n, $F_Y(y) = P(X - u_n \leq y|X > u_n)$, for $0 \leq y < x_F - u_n$, with $x_F = \sup\{x \in \mathrm{R}, F(x) < 1\}$ (upper-end point of F) generally $x_F = +\infty$. It follows from Equation (2.1) that

$$\bar{F}_Y(y) = \frac{\bar{F}(y+u_n)}{\bar{F}(u_n)} = \left(1 + \frac{y}{u_n}\right)^{-\frac{1}{\xi}} \frac{1 + (u_n + y)^{-\delta}\mathrm{L}(u_n + y)}{1 + u_n{}^{-\delta}\mathrm{L}(u_n)}$$

and if $\beta = \beta(u_n) = u_n\xi$, then $\bar{F}_Y(y)$ is a perturbed GPD distribution, where the CDF of the generalised Pareto distribution (GPD) has the form

$$G_{\beta,\xi}(y) = \begin{cases} 1 - (1 + \frac{\xi}{\beta}y)^{-\frac{1}{\xi}} & \xi \neq 0 \\ 1 - e^{-\frac{y}{\beta}} & \xi = 0 \end{cases}, y \in \begin{cases} [0, +\infty) & \xi \geq 0 \\ [0, \frac{\beta}{\xi}) & \xi < 0 \end{cases}.$$

The POT method is based on Balkama et al. (1974) result which says that the distribution of the excesses over a fixed threshold is approximated by the generalized Pareto distribution (GPD). This means that for large values of u_n, we have

$$\lim_{u_n \to x_F} \sup_{0 \leq x < x_F - u_n} |F_Y(x) - G_{\xi,\beta}(x)| = 0,$$

See also Theorem (3.4.13) in Embrechts et al (1997).

Let $X_1, ..., X_n$ be a random sample of size n. The tail of the distribution F assumed to start at some level u_n supposed sufficiently high, and the exceedance $Y_i = X_i - u_n$, for all i such that $X_i > u_n$ are approximately a random sample from a GPD. It is clear that F_Y is also regularly varying at infinity with the same index $-1/\xi < 0$.

The estimation of F_Y based on over a peaks threshold value u_n, requiring thus the choice of a threshold at which the tail of the underlying distribution begins.

If the chosen threshold is too low, the GPD approximation may not hold and bias can occur. If the threshold is chosen too high, reduced sample size increases the variance of parameter estimates. The task is to find the lowest threshold such that the GPD fits the sample of exceedances over this threshold adequately.

The problem of choosing the threshold u_n is still of high theoretical and practical interest. It is desirable to have an intuitive automated threshold selection procedure to use with POT analysis. The simple method is an a priori, or fixed threshold selection based on expertise on the subject matter at hand. Various rules have been suggested, for example, selecting the top 10% of the data, see, e.g., Dumouchel (1983), or the top 5%, see, Kelly et al. (2014), or the top square root of the sample size see, e.g., Ferreira et al. (2003), Bader et al. (2018), and Silva lomba et al. (2020).

Many threshold selection methods are available in the literature, see Dekkers (1989), Scarrott et al (2012), Gomes et al. (2008), Guillou et al. (2001), Matthys et al. (2000), and Caeiro et al. (2016) for recent reviews. The classical method is the empirical mean residual life plot (MRLP), has been used as far back as the third century (see, e.g, Davison et al. (1990) and Guess et al. (1988). Drees et al (2000) suggested the Hill plot, which plots the Hill estimator of the shape parameter based on the top k order statistics against the threshold u. Many variants of the Hill plot have been proposed (Scarrott et al, 2012). Other selection methods can be grouped into two categories.

The first is based on the asymptotic results about estimators of properties of the tail distribution. The threshold is selected by minimizing the asymptotic mean squared error (MSE) of the estimator, for example, tail index (Beirlant et al., 1996). The second category of methods are based on Goodness-of-fit of the GPD, where the threshold is selected as the lowest level above which the GPD provides adequate to the exceedance.

Those interested in extreme value theory and its applications are referred to the textbooks of de Haan et al. (2006), Embrechts et al (1997), and Beirlant et al. (2004).

3.2 POT based estimator of the ruin probability

If $X_1, X_2, ...$ be positive i.i.d RV's with CDF F given in Equation (2.1) and $u_n = O_+(n^{\alpha\xi})$ for some $\alpha \in (1/(1+2\delta\xi), 1)$, where $\delta > 0$, and if $Y_i = X_i - u_n$, for all i such that $X_i > u_n$ be positive i.i.d. RV's with CDF F_Y.

By definition

$$\bar{F}(y + u_n) = \bar{F}_Y(y)\bar{F}(u_n), \tag{3.4}$$

for $p_n = P(X_1 > u_n) = \bar{F}(u_n)$, the estimation of p_n may be done using

$$\hat{p}_n = \widehat{\bar{F}}(u_n) = \frac{1}{n}\sum_{i=0}^{n} 1_{\{X_i > u_n\}} = \frac{N_{u_n}}{n},$$

$N := N_{u_n} = \sum_{i=1}^{n} \mathbf{1}_{\{X_i > u_n\}}$, where N is the number of X_i's which exceed u_n, we have a binomial distribution, i.e., $N \rightsquigarrow \mathcal{B}(n, p_n)$.

When $np_n = n^{1-\alpha}O_+(1)$, it follows that from the Central Limit Theorem that,

$$\frac{\sqrt{n}}{\sqrt{p_n(1-p_n)}}(\hat{p}_n - p_n) \xrightarrow{D} \mathcal{N}(0,1) \text{ as } n \uparrow +\infty.$$

Further, let $\mu_n = E(X_1 1_{\{X_1 \leq u_n\}})$ with its empirical estimator $\hat{\mu}_n = n^{-1}\sum_{i=1}^{n} X_i \mathbf{1}_{\{X_i \leq u_n\}}$

and $\sigma_n^2 = Var(X_1 1_{\{X_1 \leq u_n\}})$. Then, from the Central Limit Theorem, we have,

$$\frac{\sqrt{n}}{\sigma_n}(\hat{\mu}_n - \mu_n) \xrightarrow{D} \mathcal{N}(0,1) \text{ as } n \uparrow +\infty$$

(see, e.g., Johansson (2003)). For large values of u_n, we use $\widehat{\bar{F}}_Y(y) \approx \bar{G}_{\hat{\xi}_n, \hat{\beta}_n(u_n)}(y)$, for appropriate estimates $\hat{\xi}_n$, and $\hat{\beta}_n(u_n)$. Note that β_n will be estimated separately, i.e. $\beta_n = \xi u_n$ will be note used, see, e.g., Johansson (2003).
Notice that the maximum likelihood estimates $\hat{\xi}_n$ and $\hat{\beta}_n$ are consistent and converges in probability, see, e.g., Beirlant et al. (1989) and Section 5.2, Beirlant et al. (2001).

For $\xi > -0.5$ it can be shown that maximum likelihood regularity conditions are fulfilled and that maximum likelihood estimates $(\hat{\xi}_n, \hat{\beta}_n)$ based on a sample of N excesses of a threshold u_n are asymptotically normally distributed, see e.g., Smith (1987), and Pirouzi et al (2013) as follows

$$\sqrt{np_n} Q^{\frac{1}{2}} \begin{pmatrix} \hat{\beta}_n - \beta \\ \hat{\xi}_n - \xi \end{pmatrix} \xrightarrow{D} \mathcal{N}(0, I) \ , \text{ as } n \uparrow +\infty,$$

where

$$Q^{-1} = (1+\xi) \begin{pmatrix} 2\beta^2 & -\beta \\ -\beta & 1+\xi \end{pmatrix}.$$

Under the assumption that $\sqrt{np_n} u_n \mathrm{L}(u_n) \to 0$ and that $x^{-\delta}\mathrm{L}(x)$ is non-increasing, this condition is met if $u_n = O_+\left(n^{\alpha\xi}\right)$ with $\alpha > 1/(1+2\delta\xi)$, where L is slowly varying at $+\infty$, (see Johansson (2003)). If we use the Equation (3.4), we can write

$$\int_{u_n}^{+\infty} \bar{F}(x)dx = \int_0^{+\infty} \bar{F}(y+u_n)dy = p_n \int_0^{+\infty} \bar{F}_Y(y)dy = p_n \mu_Y.$$

Then, the simplified form is

$$\Phi(u) \simeq p_n \frac{\mu_Y}{(\rho - \mu_X)}.$$

For large values of u_n, and under the assumption $\rho = \dfrac{p}{\lambda} > \mu_X$.

By the replacement of the μ_X and μ_Y by its empirical estimators, $\hat{\mu}_X$ and $\hat{\mu}_Y$ respectively, we can define an empirical estimator of $\Phi(u_n)$ as follows

$$\hat{\Phi}_n(u_n) = \hat{p}_n \frac{\hat{\mu}_Y}{(\rho - \hat{\mu}_X)}$$

For $\xi \in (0, 1/2)$, X_1 has finite variance ($\sigma_n^2 < \infty$), in this case $\mu_X = E(X_1)$ and $\mu_Y = E(Y_1)$ is estimated by the sample mean $\hat{\mu}_X = \bar{X} = n^{-1}\sum_{i=1}^n X_i$, and $\hat{\mu}_Y = \bar{Y} = n^{-1}\sum_{i=1}^n Y_i$ respectively. The estimator of $\Phi_n(u_n)$ become as follows:

$$\hat{\Phi}_n^{(1)}(u_n) = \hat{p}_n \frac{\bar{Y}}{\left(\rho - \bar{X}\right)}.$$

Using the central limit theorem, the asymptotic normality of $\hat{\Phi}(u_n)$ is established in the Theorem 4.1.

An alternative way to estimate μ_X may be made by GPD's approximation (see Johansson (2003)). Indeed, for each $n \geq 1$, we have

$$\mu_X = \int_0^\infty x dF(x) = \mu_n^* + \kappa_n,$$

where

$$\mu_n^* = \int_0^{u_n} x dF(x) \text{ and } \kappa_n = \int_{u_n}^\infty x dF(x).$$

For $\xi \in (1/2, 1)$, X_1 has an infinite variance ($\sigma_n^2 = \infty$) Johansson's estimator (see, Johansson (2003)) of μ_X is given by

$$\hat{\mu}_{X,n}^J = \widehat{\mu}_n^* + \widehat{\kappa}_n,$$

where

$$\hat{\mu}_n^* = \frac{1}{n}\sum_{i=1}^n X_i \mathbf{1}_{\{X_i \leq u_n\}} \text{ and } \widehat{\kappa}_n = \widehat{p}_n \left(u_n + \frac{\widehat{\beta}_n}{1 - \widehat{\xi}_n} \right),$$

and μ_Y is estimated by the approximate of the tail distribution function $\bar{F}$ by the GPD distribution $\bar{G}$, and after the integrating, we obtain the formula

$$\hat{\mu}_{Y,n} = \frac{\hat{\beta}_n}{1 - \hat{\xi}_n}.$$

Then, the estimator of $\Phi_n(u_n)$ become as follows:

$$\hat{\Phi}_n^{(2)}(u_n) = \hat{p}_n \frac{\hat{\mu}_{Y,n}}{\left(\rho - \hat{\mu}_{X,n}^J\right)}.$$

In the end, an asymptotic estimator of $\Phi_n(u_n)$ for any $0 < \xi < 1$ has the following form:

$$\hat{\Phi}_n(u_n) := \begin{cases} \hat{\Phi}_n^{(1)}(u_n) \text{ for } 0 < \xi \leq 1/2 \\ \hat{\Phi}_n^{(2)}(u_n) \text{ for } 1/2 < \xi < 1 \end{cases}.$$

In the following section we present our main results.

4. Main results

In this section, we present the main results of this paper. The three theorems provides the consistency and the asymptotic normality of our estimator.

Our main first result is the asymptotic normality of $\hat{\Phi}_n(u_n)$, when $E(X) < \infty$ and $E(X^2) < \infty$ (i.e., $0 < \xi \leq 1/2$). This result is a straight application of the central limit theorem.

Theorem 4.1 Let F be a CDF fulfilling Equation (2.1) with $\xi \in (0, 1/2)$, we have

$$\sqrt{n}(\hat{\Phi}_n^{(1)}(u_n) - \Phi(u_n)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2), \quad as \quad n \uparrow +\infty.$$

where the asymptotic variance σ^2 is given by the formula

$$\sigma^2 = \frac{1}{(\rho - \mu_X)^4}(\sigma_1^2(\rho - \mu_2 - u_n)^2 + \sigma_2^2(\mu_1 - u_n)^2),$$

and

$$\mu_1 = E(\bar{X}1_{\{X_i \geq u_n\}}), \quad \mu_2 = E(\bar{X}1_{\{X_i \leq u_n\}}), \quad \sigma_1^2 = Var(\bar{X}1_{\{X_i \geq u_n\}}), \quad \sigma_2^2 = var(\bar{X}1_{\{X_i \leq u_n\}}).$$

The second main result is the almost pointwise sure convergence and asymptotic normality of $\hat{\Phi}_n(u_n)$, when $E(X^2) = \infty$ (i.e., $1/2 < \xi < 1$). In this case, the central limit theorem is not applicable.

Theorem 4.2 Let F be a CDF fulfilling Equation (2.1) with $\xi \in (1/2, 1)$. Suppose that L is locally bounded in $[x_F, +\infty)$ for $x_F \geq 0$ and $x \mapsto x^{-\delta}\mathrm{L}(x)$ is non-increasing near infinity, for some $\delta > 0$. For any $u_n = O_+(n^{\alpha\xi})$ with $\alpha \in (1/(1 + 2\delta\xi), 1)$, we have

$$\Phi(u_n) = \hat{\Phi}_n^{(2)}(u_n) + o_P(1), \qquad as \ \ n \uparrow +\infty.$$

Theorem 4.3 Let F be as in Theorem 4.2, then for any $u_n = O(n^{\alpha\xi})$ with $\alpha \in (1/(1 + 2\delta\xi), 1)$, we have

$$\frac{\sqrt{n}}{\sigma}\left(\hat{\Phi}_n^{(2)}(u_n) - \Phi(u_n)\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1), \qquad as \ \ n \uparrow +\infty,$$

where

$$\sigma^2 = \theta_1 \left(p_n^2\sigma_n^2 + p_n\left(1 - p_n\right)\theta_2 + \frac{p_n(2\xi^2 - \xi + 1)(1 + \xi)}{\left(1 - \xi\right)^2}\theta_3\right) = O_+(1)$$

with

$$\theta_1 = \frac{\beta^2}{(1 - \xi)^2\left(\rho - \mu_X\right)^4}, \quad \theta_2 = \left(\rho - u_n\right)^2, \quad \theta_3 = \left(\rho - \mu_n - p_n u_n\right)^2.$$

The proofs of the above theorems are given in the Appendix.

5. Simulation Study and Application

5.1 Simulation Study

In this section, we have applied the result of the Theorem 4.2 and Theorem 4.3 we first begin to fix a significance level $\eta \in (0, 1)$ and calculate $z_{\eta/2}$ for the $(1 - \eta/2)$ quantile of the standard normal distribution $\mathcal{N}(0, 1)$. If we use a realization of the random variables $X_1, ..., X_n$, which follow a CDF F satisfying the conditions of the Theorem 4.3, we construct a level $1-\eta$ confidence interval for $\Phi(u)$ as follows.

$$\Phi(u_n) \in \left[\bar{\hat{\Phi}}_n^{(2)}(u_n) - \frac{1}{\sqrt{n}} z_{\frac{\eta}{2}} \hat{\sigma}_{\bar{\hat{\Phi}}_n^{(2)}(u_n)} \; , \; \bar{\hat{\Phi}}_n^{(2)}(u_n) + \frac{1}{\sqrt{n}} z_{\frac{\eta}{2}} \hat{\sigma}_{\bar{\hat{\Phi}}_n^{(2)}(u_n)}\right],$$

where $\bar{\hat{\Phi}}_n^{(2)}(u_n)$ and $\hat{\sigma}^2_{\bar{\hat{\Phi}}_n^2(u_n)}$ are the empirical mean and variance respectively.

We realized a simulation study to validate the performance of the ruin probability estimation and the asymptotic normality of our proposed estimator. All numerical evaluations and graphics presented here were done in the `R` software using the packages `POT`, `evir`, `extremefit`, and actuar for Extreme values. To this end, we simulated samples of the Pareto, Burr and Log-Gamma distributions, whose tail index parameters are summarized in Table 1.

Table 1. Regularly varying distribution functions

Distribution	Tail $\bar{F}(x)$ or density $f(x)$	Parameters	Index Parameter $(1/\xi)$
Pareto	$\bar{F}(x) = x^{-c}$, $x > 1$	$c > 0$	c
Burr	$\bar{F}(x) = (1 + x^c)^{-k}$, $x > 0$	$c, k > 0$	ck
Log-Gamma	$f(x) = c^k/\Gamma(k)(\ln(x))^{k-1}x^{-c-1}$, $x > 0$	$c, k > 0$	c

We generated 200 samples of sizes 500, 1000 and 5000 from the previous distributions with the index values $\xi = 2/3$ and $\xi = 3/4$.
For each simulated sample, we obtain a value of the estimators $\hat{\Phi}_n^{(2)}(u_n)$. The overall estimated $\hat{\Phi}_n^{(2)}(u_n)$ is then taken as the empirical mean of the values in the 200 repetitions with its confidence interval. We also obtain the bias and the root mean squared error (RMSE) of the estimator of $\hat{\Phi}_n^{(2)}(u_n)$. We summarize the results in Table 2 and Table 3.

Table 2. 95% confidence interval for the ruin probability of Pareto, Burr and Log-Gamma Distributions with tail index $\xi = 2/3$

Distribution	n	$\bar{\hat{\Phi}}(u)$	Bias	RMSE	Confidence Interval
		$\Phi(u) = 0.08186737$			
Pareto	500	0.079678	-0.002188	0.027843	(0.073641 , 0.085715)
	1000	0.081419	-0.000447	0.022333	(0.077098, 0.08574157)
	5000	0.081760	0.000106	0.020226	(0.080826, 0.082695)
		$\Phi(u) = 0.065366$			
Burr	500	0.061051	-0.004314	0.008490	(0.051344 , 0.070759)
	1000	0.061616	- 0.003749	0.007504	(0.056035 , 0.067198)
	5000	0.064166	-0.001199	0.006906	(0.061623, 0.066710)
		$\Phi(u) = 0.07368063$			
Log-Gamma	500	0.071909	-0.001771	0.025905	(0.059811, 0.084006)
	1000	0.075159	0.001479	0.023896	(0.072138, 0.078181)
	5000	0.074933	0.001253	0.019920	(0.072138 , 0.077729)

Table 3. 95% confidence interval for the ruin probability of Pareto, Burr and Log-Gamma Distributions with tail index $\xi = 3/4$.

Distribution	n	$\bar{\hat{\Phi}}(u)$	Bias	RMSE	Confidence Interval
		$\Phi(u) = 0.177327$			
Pareto	500	0.161809	-0.015517	0.054281	(0.141244 , 0.182373)
	1000	0.168686	-0.008639	0.044726	(0.154511, 0.182861)
	5000	0.179787	0.002460	0.036559	(0.176026 ,0.183547)
		$\Phi(u) = 0.1052539$			
Burr	500	0.094451	-0.010802	0.007272	(0.057324 , 0.122166)
	1000	0.099181	-0.006072	0.004516	(0.082715,0.115647)
	5000	0.100817	-0.004436	0.004436	(0.094665, 0.106969)
		$\Phi(u) = 0.1576236$			
Log-Gamma	500	0.171419	0.013795	0.053793	(0.150337 , 0.192500)
	1000	0.159431	0.001807	0.038889	(0.150499 , 0.168363)
	5000	0.157925	0.000302	0.035215	(0.155589 , 0.160261)

5.2 Application

We illustrate an application to the Danish fire insurance data (in millions DKK). The Danish data on large fire insurance claims are widely used and provides an exceptional example of the use of the extreme value theory in a significant application context, see McNeil (1996), Mikosch (2006). The full Danish data of Reinsurance comprise 2167 fire losses from Thursday 3rd January 1980 until Monday 31st December 1990. Mikosch (Mikosch, 2006) confirm that a homogeneous Poisson process is an appropriate model for the arrivals of the Danish fire insurance data for shorter periods of time such as one year with the parameter $\lambda = 1/1.85$. McNeil (McNeil, 1996) was adjusted the danish fire insurance data, he concluded that the Lognormal distribution to be good, but the Pareto distribution did not fit the data well, while the GPD distribution was acceptable. We restrict our attention to the 2156 losses exceeding one million. The descriptive statistical study of the Danish data is summarized in Table 4.

Table 4. Descriptive Statistics Summary of the dataset of Danish fire

min	Mean	Std	First Quartile	Median	Third Quartile	max
1	1.85	8.5274	1.321	1.778	2.967	263.250

All numerical evaluations and graphics presented here were done in the `R` software using the packages `POT`, `evir`, `extremefit`, and actuar for Extreme values. We also used the functions `qqnorm` for the adjustment with the normal distribution. For the resampling, we used the package `boot`. The generalized Pareto distribution can be fitted to data on excesses of high thresholds by a variety of methods including the maximum likelihood method (ML) to estimate the parameters and the the Mean Residual Life Plot (MRLP) to select the threshold. The MRLP function (see again Figure 2) is the plot of $\{(u_n, e_n(u_n)), X_{(1)} < u_n < X_{(n)}\}$ where $X_{(1)}$, and $X_{(n)}$ are the first and nth order statistics and $e_n(u)$ is the sample mean excess function gives by the sum of the excesses over the threshold u_n divided by the number of data points which exceed the threshold u_n.

$$e_n(u_n) = \frac{1}{N}\sum_{i=1}^{N}(X_i - u_n), X_i > u_n.$$

The interpretation of the mean excess plot as explained in Beirlant et al. (1996) and Embrechts et al (1997). There is evidence of a straightening out of the plot above a threshold of 1, 10 and perhaps 20 (See Figure 2).

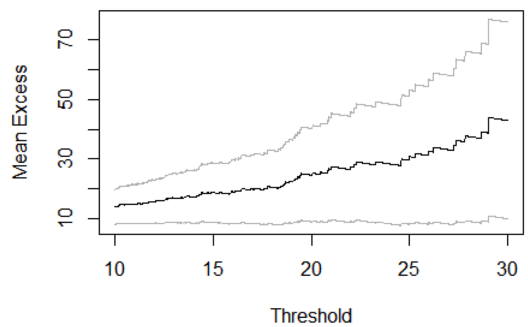


Figure 2. Mean Residual Life plot.

The points show an upward trend, then this is a sign of heavy tailed behaviour. The Histogram (Figure 3(a)) shows that the data may perhaps is a heavy tailed distribution.

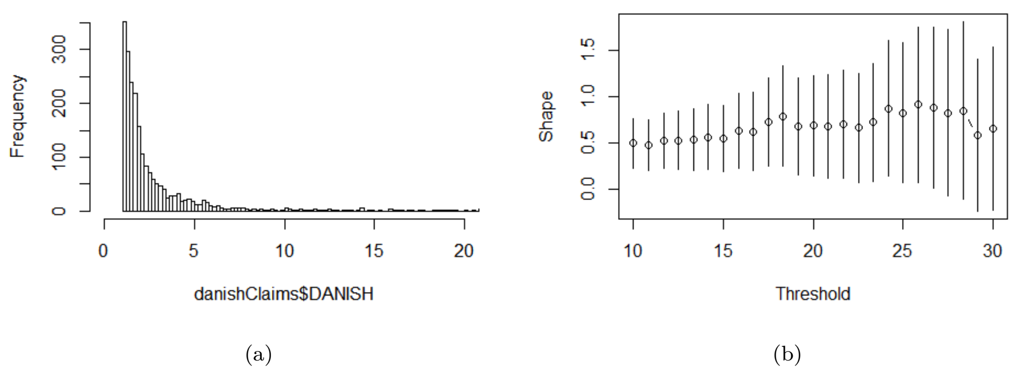


Figure 3. The histogram (a) and shape parameter as function of the threshold (b) for the Danish data.

It is Noted that we must take the threshold $u \geq 20$ to have the shape parameter $\xi > 0.5$, which fits our case (see again Figure 3(b)).

We look at standard choices of curve fitted to the whole dataset. We use the GPD with the threshold $u = 20$, the maximum likelihood estimators of the parameters are given by the shape $\hat{\xi} = 0.6840479$ and the scale $\hat{\beta} = 9.6316941$. The parameter estimates are the same as those given by McNeil (1996) for the GPD distribution. The QQ-plot in Figure 4 indicate that the ruin probability of the Danish data via bootstrap sampling (random sampling with replacement), see, e.g., Bukhalter et al. (2021) is well fitted by the normal distribution when the losses exceeding a threshold $u = 20$.

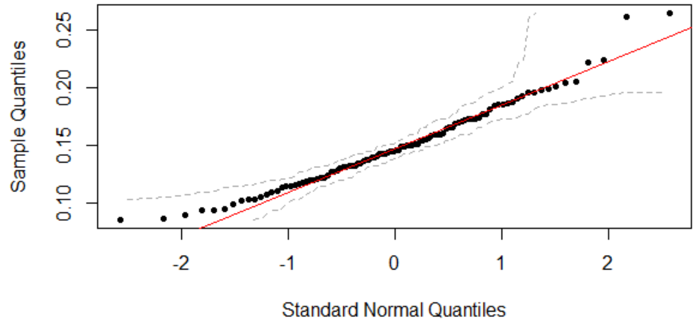


Figure 4. The QQ-plot for ruin probability with bootstrap method.

Conclusions, limitations, and future research

In this paper, we have proposed a new estimator for the ruin probability of heavy-tailed claims amounts via semi-parametric estimation. Our approach is based on a variant of extreme value theory, called the Peaks Over the Threshold method. We have demonstrated the consistency and the normality of the estimator of the ruin probability. Finally, we have validated these results with a simulation study. We calculated the ruin probability for danish fire insurance claims and used the bootstrap method to prove that the last is normal. Our proposal provides interesting results and constitutes a tool that can be useful for actuarial researchers and insurance companies when modeling for instance fire, and storm damages. Our proposal has some limitations such as the choice of the threshold which remains a theoretical and practical research problem, and we must have a large sample of claims and large initial capital. This limitations open some doors for further research, which will be considered by the authors in future works.

Author contributions Conceptualization, F.R., R.A; Methodology, R.A., F.R; Software, F.R; Validation, R.A, O.H; Formal analysis, R.A, O.H.; Investigation, F.R; Data Curation, F.R, R.A; Visualization, F.R, R.A; Supervision, R.A, O.H; Writing-original draft, F.R; Writing-review and editing, R.A, O.H.

All authors have read and agreed the published version of the manuscript.

Acknowledgements

We thank all the corrections and suggestions pointed out by the Editors-in-Chief and the reviewers.

Funding The authors received no financial support for the research, authorship, and/or publication of this article.

Conflicts of interest The authors declare no conflict of interest.

References

Abubakar, S.A., Hamzah, N., Maghsoudi, M., and Nadarajah, S., 2015. Modeling loss data using composite models. Insurance: Mathematics and Economics 64, 146-154.

Asmussen, S., 2010. Ruin Probabilities. World Scientific, Singapore, New Jersey, London, Hong Kong.

Aurzada, F., Buck, M., 2020. Ruin probabilities in the Cramér-Lundberg model with temporarily negative capital. European Actuarial Journal, 10, 261-269.

Balkema, A., de Haan, L., 1974. Residual Life Time at Great Age. The Annals of Probability.

Bader, B., J. Yan, and X. Zhang, 2018. Automated threshold selection for extreme value analysis via ordered goodness-of-fit tests with adjustment for false discovery rate. Ann. Appl. Stat. 12, 310-29.

Behme, A., Klüppelberg, C., and Reinert, G., 2020. Ruin probabilities for risk processes in a bipartite network. vol. 0, Taylor & Francis, pp 1-26.

Beirlant, J., Vynckier, P., and Teugels, J.L., 1996. Tail Index Estimation, Pareto Quantile Plots, and Regression Diagnostics. Journal of the American Statistical Association, 91, 1659-1667.

Beirlant, J., Matthys, G., and Dierckx, G., 2001. Heavy-tailed distributions and rating. Astin Bull. 31, 37-58.

Beirlant, J., Teugels, J., 1989. Asymptotic normality of Hill's estimator. In: Extreme Value Theory (Oberwolfach, 1987). Lecture Notes in Statistics, vol. 51, Springer, New York, pp 148-155 .

Beirlant, J., Goegebeur,Y., Teugels,J., and Segers, J., 2004. Statistics of Extremes. Wiley Series in Probability and Statistics, John Wiley & Sons, Chichester, UK.

Burkhalter, R. and Lio, Y., 2021. Bootstrap control charts for the generalized Pareto distribution percentiles. Chilean Journal of Statistics 12(1), 3-21.

Caeiro, F., and Gomes, M. I., 2016. Threshold Selection in Extreme Value Analysis. Extreme Value Modeling and Risk Analysis: Methods and Applications, 69-82.

Concepcion, A., and Hedibert, L., 2007. Bayesian estimation of ruin probabilities with heterogeneous and heavy-tailed insurance claim size distribution. Australian & New Zealand Journal of Statistics, 415-434.

Cramer, H., 1930. On the Mathematical Theory of Risk. Försäkringsaktiebolaget Skandias Festskrift. Stockholm: Centraltryckeriet, 7-84. [Google Scholar]

Davison, A.C. and Smith, R.L., 1990. Models for Exceedances Over High Thresholds. Journal of the Royal Statistical Society, Series B (Methodological), 52, 393-442.

de Haan, L., and Ferreira, A., 2006. Extreme Value Theory: An Introduction, Springer Series in Operations Research and Financial Engineering. Springer, New York, USA.

Dekkers, A., Einmahl, J., and de Haan, L., 1989. A moment estimator for the index of an extreme-value distribution. Ann. Statist. 17, 1833-1855.

Dekkers, A., and de Haan, L., 1989. On the estimation of the extreme value index and large quantile estimation. Ann. Statist. 17, 1795-1832.

Dekkers, A., and de Haan, L., 1993. Optimal choice of sample fraction in extreme-value estimation. J. Mult. Anal. 47, 173-195.

De Vylder, F.E. (1978). A practical solution to the problem of ultimate ruin probability. Scandinavian Actuarial Journal, 114-119.

Drees, H., de Haan, L., and Resnick, S., 2000. How to Make a Hill Plot. The Annals of Statistics, 28, 254-274.

DuMouchel, W. H., 1983. Estimating the Stable Index α in Order to Measure Tail Thickness: A Critique. The Annals of Statistics, 11, 1019-1031.

Ferreira, A., de Haan, L., and Peng, L., 2003. On optimising the estimation of high quantiles of a probability distribution. Statistics: A Journal of Theoretical and Applied Statistics

37(5), 401-434.
Embrechts, P., Klüppelberg, C., and Mikosch, T., 1997. Modelling Extremal Events. Springer, Berlin, Germany.
Gomes, M. I., Henriques-Rodrigues, L., Vandewalle, B., and Viseu, C., 2008. A heuristic adaptive choice of the threshold for bias-corrected Hill estimators. Journal of Statistical Computation and Simulation, 78, 133-150.
Guess,F. and Proschan, F., 1988. Mean residual life: Theory and applications, Handbook of Statistic, 8, 215-224.
Guillou, A., and Hall, P., 2001. A diagnostic for selecting the threshold in extreme value analysis. J. R. Statist. Soc. B, 63, 293-305.
Johansson, J., 2003. Estimating the mean of heavy-tailed distributions. Extremes, 6, 91-131.
Kelly, B., and Jiang, H., 2014. Tail risk and asset prices. Review of Financial Studies, 27(10), 2841-2871.
Klüppelberg, C., and Stadtmüller, U., 1996. Ruin Probabilities in the Presence of Heavy-Tails and Interest Rates. Scand. Actuar. J, 49, 49-58.
Lundbergh, F., 1903. Approximerad framställning af sannolikhetsfunktionen: Aterförsäkering af kollektivrisker. PhD thesis, Almqvist & Wiksell.
Matthys, G., and Beirlant, J., 2000. Adaptive threshold selection in tail index estimation, in Extremes and Integrated Risk Management. ed. Embrechts, P., London, UK: Risk Books, 37-49.
Mikosch, T., and Samorodnitsky, G., 2000. Ruin probability with claims modeled by a stationary ergodic stable process. The Annals of Probability, 28(4), 1814-1851.
Mikosch, T., 2006. Non-Life Insurance Mathematics. An Introduction with Stochastic Processes. Springer-Verlag Berlin Heidelberg New York, 28-32.
McNeil, A., 1996. Estimating the Tails of Loss Severity Distributions Using Extreme Value Theory. ASTIN Bulletin, 27, 117-137.
Pirouzi, M., Fard, M., and Holmquist, B., 2013. Powerful goodness-of-fit tests for the extreme value distribution. Chilean Journal of Statistics, 4(1), 55-67.
Reiss, R., and Thomas, M., 2002. Statistical Analysis of Extreme Values with Application to Insurance, Finance, Hydrology and other fields (Third Edition).
Resnick, S., 1987. Extreme Values, Regular Variation and Point Processes. Springer-Verlag.
Scarrott, C., and MacDonald, A., 2012. A review of extreme value threshold estimation and uncertainty quantification. REVSTAT-Stat J; 10(1), 33-60.
Seneta, E., 1976. Regularly Varying Functions, no. 508 in Lecture Notes in Mathematics, Springer-Verlag.
Silva, L., and Fraga Alves, M., 2020. L-moments for automatic threshold selection in extreme value analysis. Stochastic Environmental Research and Risk Assessment, 34, 465-491.
Smith, R., 1987. Estimating Tails of Probability Distributions. The Annals of Statistics, 15, 1174-1207.
Sundt, B., and Teugels, J., 1995. Ruin estimates under interest force. Insurance Math.Econom. 16, 7-22.
Xiaoping, Z., and Huang, C., 2014. Ruin Probabilities in the Mixed Claim Frequency Risk Models. Hindawi Publishing Corporation, Mathematical Problems in Engineering, 314307, 1.7.
Vladimir, K., and Dimitrios, K., 2000. Ruin under interest force and subexponential claims: a simple treatment. Insurance: Mathematics and Economics, 27, 145-149.
Zhu, L., 2013. Ruin probabilities for risk processes with non-stationary arrivals and subexponential claims, Insurance: Mathematical and Economics, 53, 544-550.

Appendix

Proof of Theorem 4.1

Proof The proof of this theorem is based on the multivariate Delta method theorem who generalise the central limit theorem.

If we take

$$Z_n = \begin{pmatrix} Z_n^1 \\ \\ Z_n^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^n X_i 1_{\{X_i \geq u_n\}} \\ \\ \frac{1}{n}\sum_{i=1}^n X_i 1_{\{X_i \leq u_n\}} \end{pmatrix} = \begin{pmatrix} \bar{X} 1_{\{X_i \geq u_n\}} \\ \\ \bar{X} 1_{\{X_i \leq u_n\}} \end{pmatrix}.$$

Therefore the asymptotic normality of Z_n follows directly from the classical Central Limit Theorem

$$\sqrt{n}(Z_n - \mu) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma), as \quad n \uparrow +\infty,$$

where $\mu = \begin{pmatrix} \mu_1 \\ \\ \mu_2 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ \\ 0 & \sigma_2^2 \end{pmatrix}$, and if we take $g(s_1, s_2) = s_1 - u_n/\rho - (s_1 + s_2)$, we obtain

$$\sqrt{n}(g(Z_n) - g(\mu)) = \sqrt{n}\left(\hat{p}_n \frac{\bar{Y}}{\left(\rho - \bar{X}\right)} - p_n \frac{\mu_Y}{\rho - \mu_X}\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \triangledown g(\mu)^T \Sigma \triangledown g(\mu)\right), as \quad n \uparrow +\infty.$$

where

$$\triangledown g(\mu) = \begin{pmatrix} \frac{\partial g}{\partial y_1}(\mu) \\ \frac{\partial g}{\partial y_2}(\mu) \end{pmatrix}$$

and

$$\sigma^2 := \triangledown g(\mu)^T \Sigma \triangledown g(\mu) = \frac{1}{(\rho - \mu_X)^4}\left(\sigma_1^2(\rho - \mu_2 - u_n)^2 + \sigma_2^2(\mu_1 - u_n)^2\right).$$

■

Proof of Theorem 4.2

Proof Let us write

$$\begin{aligned}
\hat{\Phi}_n^{(2)}(u_n) - \Phi(u_n) &= \left(\hat{p}_n \frac{\hat{\mu}_{Y,n}}{\left(\rho - \hat{\mu}_{X,n}^J\right)} - p_n \frac{\mu_Y}{(\rho - \mu_X)}\right) \\
&= \frac{(\hat{p}_n \hat{\mu}_{Y,n} - p_n \mu_Y)(\rho - \mu_X) + \left(\hat{\mu}_{X,n}^J - \mu_X\right) p_n \mu_Y}{\left(\rho - \hat{\mu}_{X,n}^J\right)(\rho - \mu_X)}.
\end{aligned}$$

If we note $A_n = \left(\rho - \hat{\mu}_{X,n}^J\right)(\rho - \mu_X)\left(\hat{\Phi}(u_n) - \Phi(u_n)\right)$.

For GPD's approximation, Johansson has proposed (see, Johansson (2003)) the estimators of $\hat{\mu}^J_{X,n}$ and $\hat{\mu}_Y$ as follows

$$\hat{\mu}^J_{X,n} = \hat{\mu}^*_n + \hat{\kappa} = \hat{\mu}^*_n + \hat{p}_n \left(u_n + \frac{\hat{\beta}}{1-\hat{\xi}} \right)$$

and $\hat{\mu}_Y = \frac{\hat{\beta}}{1-\hat{\xi}}$ (Y has a GPD distribution). This expression may be rewritten as follows

$$\begin{aligned} A_n &= \left(\frac{\hat{p}_n \hat{\beta}_n}{\left(1-\hat{\xi}_n\right)} - \frac{p_n \beta}{1-\xi} \right) \left(\rho - \mu - p_n \left(u_n + \frac{\beta}{1-\xi} \right) \right) \\ &\quad + \left(\hat{\mu} + \hat{p}_n \left(u_n + \frac{\hat{\beta}_n}{\left(1-\hat{\xi}_n\right)} \right) - \mu - p_n \left(u_n + \frac{\beta}{1-\xi} \right) \right) \frac{p_n \beta}{1-\xi}. \end{aligned}$$

After rearrangement, we can write

$$\begin{aligned} A_n = &\left(\frac{\hat{p}_n \hat{\beta}_n}{\left(1-\hat{\xi}_n\right)} - \frac{p_n \hat{\beta}_n}{\left(1-\hat{\xi}_n\right)} + \frac{p_n \hat{\beta}_n}{\left(1-\hat{\xi}_n\right)} - \frac{p_n \beta}{1-\xi} \right) \left(\rho - \mu_n - p_n \left(u_n + \frac{\beta}{1-\xi} \right) \right) \\ &+ \left((\hat{\mu}_n - \mu_n) + (\hat{p}_n - p_n) \left(u_n + \frac{\hat{\beta}_n}{\left(1-\hat{\xi}_n\right)} \right) + p_n \left(\frac{\hat{\beta}_n}{\left(1-\hat{\xi}_n\right)} - \frac{\beta}{1-\xi} \right) \right) \frac{p_n \beta}{1-\xi} \ . \end{aligned}$$

Using the approximations of Johansson (Johansson (2003)), we can deduce

$$\begin{aligned} A_n &= \frac{p_n \beta}{1-\xi} (\hat{\mu}_n - \mu_n) + \frac{\beta}{1-\xi} (\rho - \mu_n) (\hat{p}_n - p_n) \\ &\quad + \frac{p_n(\rho - \mu_n - p_n u_n)}{(1-\xi)} \left(\hat{\beta}_n - \beta \right) \\ &\quad + \frac{p_n \beta(\rho - \mu_n - u_n p_n)}{(1-\xi)^2} \left(\hat{\xi}_n - \xi \right) + o_{\mathrm{P}}(1) \end{aligned} \tag{5.5}$$

then, we can rewrite

$$\hat{\Phi}^{(2)}_n(u_n) - \Phi(u_n) = \frac{1}{\left(\rho - \hat{\mu}^J_{X,n}\right)(\rho - \mu_X)} (A_{n,1} + A_{n,2} + A_{n,3} + A_{n,4}) + o_{\mathrm{P}}(1)$$

where

$$A_{n,1} = \frac{p_n \beta}{1-\xi} \frac{1}{\left(\rho - \hat{\mu}^J_{X,n}\right)(\rho - \mu_X)} (\hat{\mu}_n - \mu_n),$$

$$A_{n,2} = \frac{\beta}{1-\xi} (\rho - \mu_n) \frac{1}{\left(\rho - \hat{\mu}^J_{X,n}\right)(\rho - \mu_X)} (\hat{p}_n - p_n)$$

$$A_{n,3} = \frac{p_n(\rho - \mu_n - p_n u_n)}{(1-\xi)} \frac{1}{\left(\rho - \hat{\mu}^J_{X,n}\right)(\rho - \mu_X)} \left(\hat{\beta}_n - \beta\right)$$

$$A_{n,4} = \frac{p_n \beta(\rho - \mu_n - u_n p_n)}{(1-\xi)^2} \frac{1}{\left(\rho - \hat{\mu}^J_{X,n}\right)(\rho - \mu_X)} \left(\hat{\xi}_n - \xi\right)$$

We need the following proposition to complete the demonstration of the Theorem 4.2.

PROPOSITION 5.1 Let F_X be a CDF fulfilling Equation (2.1) with $\xi \in (0,1)$, $\delta > 0$ and real c. suppose that L is locally bounded in $[x_F, \infty)$ for somme $x_F \geq 0$. Then fr large n enough and for any $u_n = O(n^{\alpha\xi})$, $\alpha \in (0,1)$, we have

$$\sigma_n^2 = O_{\mathrm{P}}(n^{\alpha(2\xi-1)}),$$

$$(\hat{\mu}_n - \mu_n) = O_{\mathrm{P}}(\frac{\sigma_n}{\sqrt{n}}),$$

$$(\hat{p}_n - p_n) = O_{\mathrm{P}}(\frac{\sqrt{p_n(1-p_n)}}{\sqrt{n}}),$$

$$\left(\hat{\beta}_n - \beta\right) = O_{\mathrm{P}}(\frac{1}{\sqrt{p_n n}}),$$

$$\left(\hat{\xi}_n - \xi\right) = O_{\mathrm{P}}(\frac{1}{\sqrt{p_n n}})$$

Proof For the proof of the proposition (see, e.g.,Smith (1987)). We can deduce that

$$(\hat{\mu}_n - \mu_n) = o_{\mathrm{P}}(1), \text{as } n \uparrow +\infty,$$

$$(\hat{p}_n - p_n) = o_{\mathrm{P}}(1), \text{as } n \uparrow +\infty,$$

$$\left(\hat{\beta}_n - \beta\right) = o_{\mathrm{P}}(1), \text{as } n \uparrow +\infty,$$

$$\left(\hat{\xi}_n - \xi\right) = o_{\mathrm{P}}(1), \text{as } n \uparrow +\infty.$$

■

Under the assumption that $\rho > \mu_X$, It follows that

$$\frac{p_n \beta}{1-\xi} \frac{1}{\left(\rho - \hat{\mu}^J_{X,n}\right)(\rho - \mu_X)} (\hat{\mu}_n - \mu_n) = o_{\mathrm{P}}(1) \text{ as } n \uparrow +\infty,$$

$$\frac{\beta}{1-\xi} (\rho - \mu_n) \frac{1}{\left(\rho - \hat{\mu}^J_{X,n}\right)(\rho - \mu_X)} (\hat{p}_n - p_n) = o_{\mathrm{P}}(1) \text{ as } n \uparrow +\infty,$$

$$\frac{p_n(\rho-\mu_n-p_n u_n)}{(1-\xi)} \frac{1}{\left(\rho-\hat{\mu}^J_{X,n}\right)(\rho-\mu_X)}\left(\hat{\beta}_n-\beta\right) = o_{\mathrm{P}}(1) \text{ as } n\uparrow+\infty,$$

and

$$\frac{p_n\beta(\rho-\mu_n-u_n p_n)}{(1-\xi)^2} \frac{1}{\left(\rho-\hat{\mu}^J_{X,n}\right)(\rho-\mu_X)}\left(\hat{\xi}_n-\xi\right) = o_{\mathrm{P}}(1) \text{ as } n\uparrow+\infty.$$

Hence we have

$$\hat{\Phi}^{(2)}_n(u_n)-\Phi(u_n)=o_{\mathrm{P}}(1)$$

This completes the proof of Theorem 4.2. ■

Proof of Theorem 4.3

Proof The Lemma A.2 in Johansson (2003) gives that the estimators conditional on N, $\hat{\mu}_n$ is independent of $\left(\hat{\beta}_n,\hat{\xi}_n\right)$. And if we use that

$$\sqrt{np_n}Q^{\frac{1}{2}}\begin{pmatrix}\hat{\beta}_n-\beta\\ \hat{\xi}_n-\xi\end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}(0,I)\ , \text{ as } n\uparrow+\infty,$$

where

$$Q^{-1}=(1+\xi)\begin{pmatrix}2\beta^2 & -\beta\\ -\beta & 1+\xi\end{pmatrix},$$

$$\frac{\sqrt{n}}{\sqrt{p_n(1-p_n)}}(\hat{p}_n-p_n) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \text{ as } n\uparrow+\infty,$$

and

$$\frac{\sqrt{n}}{\sigma_n}(\hat{\mu}_n-\mu_n) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \text{ as } n\uparrow+\infty.$$

Using the result given in Equation (5.5) in the proof of the Theorem 2 and if we note $M=1/\left(\rho-\hat{\mu}^J_{X,n}\right)(\rho-\mu_X)$, we can write

$$\begin{aligned}\sqrt{n}\left(\hat{\Phi}^{(2)}_n(u_n)-\Phi(u_n)\right) = M\Big[&\frac{\sigma_n\sqrt{n}p_n\beta}{\sigma_n(1-\xi)}(\hat{\mu}_n-\mu_n)+\frac{\sqrt{n}\sqrt{p_n(1-p_n)}\beta}{(1-\xi)\sqrt{p_n(1-p_n)}}(\rho-\mu_n)(\hat{p}_n-p_n)\\ &+\frac{\sqrt{n_{p_n}}\sqrt{p_n}}{(1-\xi)}(\rho-\mu_n-p_n u_n)(\hat{\beta}_n-\beta)\\ &+\frac{\sqrt{n_{p_n}}\sqrt{p_n}\beta}{(1-\xi)^2}(\rho-\mu_n-u_{n_{p_n}})\left(\hat{\xi}_n-\xi\right)\Big]+o_{\mathrm{P}}(1).\end{aligned}$$

By the law of large numbers the random variable $1/\left(\rho-\hat{\mu}^{J}_{X,n_X}\right)(\rho-\mu_X)$ converge in probability to $1/(\rho-\mu_X)^2$. Also, by the Slutsky's Theorem, we can justify the asymptotic normality of $\sqrt{n}\left(\hat{\Phi}^{(2)}_n(u_n)-\Phi(u_n)\right)$. We can deduce the asymptotic variance as follows

$$\sigma^2=\frac{\beta^2}{(1-\xi)^2(\rho-\mu_X)^4}\Big[p_n^2\sigma_n^2+p_n(1-p_n)(\rho-u_n)^2+2p_n(1+\xi)(\rho-\mu_n-p_nu_n)^2$$
$$+\frac{p_n(1+\xi)^2}{(1-\xi)^2}(\rho-\mu_n-p_nu_n)^2-\frac{2p_n(1+\xi)}{(1-\xi)}(\rho-\mu_n-p_nu_n)^2\Big].$$

If we note

$$\theta_1=\frac{\beta^2}{(1-\xi)^2(\rho-\mu_X)^4},\quad \theta_2=(\rho-u_n)^2,\quad \theta_3=(\rho-\mu_n-p_nu_n)^2,$$

we can rewrite σ^2 as follows

$$\sigma^2=\theta_1\left(p_n^2\sigma_n^2+p_n(1-p_n)\theta_2+\frac{p_n(2\xi^2-\xi+1)(1+\xi)}{(1-\xi)^2}\theta_3\right)=O_+(1)$$

■