CHILEAN JOURNAL OF STATISTICS

Edited by Víctor Leiva and Carolina Marchant

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Web of Science Group

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Volume 11 Number 2
December 2020
ISSN: 0718-7912 (print)
ISSN: 0718-7920 (online)

Published by the
Chilean Statistical Society
SOCHE
SOCIEDAD CHILENA DE ESTADÍSTICA

Aims

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Chilean Journal of Statistics
Vol. 11, No. 2, December 2020, 95–116

Multivariate statistics
Research Paper

Inference for the trivariate Marshall-Olkin-Weibull distribution in presence of right-censored data

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(Received: 20 June 2020 · Accepted in final form: 02 December 2020)

Abstract

Multivariate lifetime data are common in many applications, especially in medical and engineering studies. In this paper, we consider a trivariate Marshall-Olkin-Weibull distribution to model trivariate data in presence of right censored data.Maximum likelihood and Bayesian methods are used to get the parameter estimators of interest. An extensive simulation study was performed to verify the effectiveness of the maximum likelihood estimators. Reliability data sets related to fiber failure strengths were considered to illustrate the performance of the proposed model under the classical and Bayesian approaches. As a result, note that the trivariate Marshall-Olkin-Weibull model could be considered as a good alternative to model trivariate lifetime data, especially under a Bayesian approach which could be of interest for the reliability analysis, as observed with the real data application in industrial engineering presented in the study or any other area of interest.

Keywords: Bayesian approach · Censored data · Maximum likelihood method · Monte Carlo simulation · Multivariate distributions.

Mathematics Subject Classification: Primary 62-XX · Secondary 62Hxx.

1. Introduction

Lifetime distributions have been studied extensively in the literature due to its medical and engineering applications. Usually it is possible to have two or more lifetimes associated with each subject as for example in medical recurrent events. In these situations, it is needed statistical models which capture the dependence among the lifetimes related to each unit. These lifetime data may be censored at a fixed time point due to the limitation of the follow-up period or withdrawal of the subject from the study. Assuming two lifetime observations, Arnold and Strauss (1988); Sarkar (1987); Hawkes (1972); Downton (1970); Gumbel (1960) introduced some bivariate distributions with exponential conditionals. Block and Basu (1974); Marshall and Olkin (1967a); Freund (1961) proposed extensions of the bivariate exponential distribution. In other direction, Basu and Dhar (1995) and Arnold (1975) introduced some bivariate geometric distributions. Pellerey (2008) modeled

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ISSN: 0718-7912 (print)/ISSN 0718-7920 (online)

http://www.soche.cl/chjs

dependent lifetimes using Archimedean survival copulas. Moreover, assuming three or more lifetimes, Gultekin and Bairamov (2013); De Oliveira et al. (2021) introduced trivariate geometric distributions. Hougaard (1986) proposed a class of multivariate failure time distributions. Marshall and Olkin (1967b) introduced a multivariate exponential distribution. Arellano-Valle and Genton (2010) introduced multivariate unified skew-elliptical distributions and Richter and Venz (2014) proposed geometric representations of multivariate skewed elliptically contoured distributions.

Considering the univariate situation, a distribution which is widely considered in the lifetime data analysis is the Weibull distribution (Weibull, 1951) given the flexibility of fit for the data. The mathematical properties and its applicability and generalizations have been studied by many authors (see for example, Cohen, 1965; Philip, 1974; Lai et al., 2003; Thoman et al., 1969; Stevens and Smulders, 1979; Rinne, 2008; Mudholkar et al., 1996; Brown and Wohletz, 1995; Pinder III et al., 1978; Cao, 2004; Pham and Lai, 2007; Saraiva and Suzuki, 2017; among many others). In this study, we explore a multivariate exponential distribution introduced by Marshall and Olkin (1967b) given as an extension of the fatal shock model to a multi-component system to build a new trivariate lifetime distribution denoted as the trivariate Marshall-Olkin-Weibull (TMOW) distribution.

We assume three lifetime random variables denoted following this new distribution in presence of right censored data. Maximum likelihood (ML) inference methods using numerical iterative techniques and Bayesian methods using Markov chain Monte Carlo (MC) methods are used to get the inferences of interest. Under the classical approach, the inferences of interest are obtained using standard asymptotically normality of the likelihood function considering the observed Fisher information matrix in place of the usual expected Fisher information matrix given the complexity of the likelihood function. An extensive simulation study is also performed to verify the effectiveness of the considered inference method assuming different fixed values for the parameters of the model and different sample sizes. An application for real data is also presented in order to verify the usefulness of the proposed model.

The paper is organized as follows: in Section 2, it is introduced the TMOW along with some mathematical properties. The estimation procedures assuming complete and censored data are introduced in Section 3 and 4. In Section 5, the results of the MC simulation study are presented to evaluate the biases, the root of the mean squared error and the asymptotic normality of the ML estimators for the TMOW distribution. Section 6 presents an application to reliability data related to fiber failure strengths. Section 7 provides some concluding remarks.

2. The TMOW distribution

The TMOW distribution is constructed considering k-independent Poisson processes governing the occurrence of shocks to components $1, \ldots, k$, respectively; governing the occurrence of shocks to components pairs 1 and 2, 1 and 3, ..., $k-1$ and k, respectively; and so on. This construction of the TMOW distribution plays a central role in life testing and reliability analysis since it has exponential marginal distributions, a useful property in many applications.

It is worth mentioning that an important property of the TMOW distribution is that it is not absolutely continuous since it has singular parts (Marshall and Olkin, 1967b)). In addition, the TMOW distribution could be also represented in terms of independent exponentials since there exist independent exponential random variables Z_s such that $X_i = \min_{s_i=1} \quad Z_s$, for $i = 1, \ldots, k$ obtained from the fatal shock model.

Let $\boldsymbol{Y} = (Y_1, \ldots, Y_k)$ be a random vector and consider the occurrence of simultaneous shocks to all k-components assuming the fatal shock model. Then, the survival function

(SF) of this special case of the TMOW distribution with $k+1$ parameters is given by

$$\begin{aligned} S(y_1,\ldots,y_k) &= \mathrm{P}(Y_1 > y_1,\ldots,Y_k > y_k) \\ &= \exp\{-\lambda_1 y_1 - \cdots - \lambda_k y_k - \lambda_{k+1}\max(y_1,\ldots,y_k)\}, \end{aligned} \tag{1}$$

where $\lambda_j > 0$ and $y_j > 0$, for $j = 1,\ldots,k+1$. Notice that the TMOW distribution is, mathematically, a fairly simple distribution, however, its marginal distributions could be inappropriate to model the behavior of units which have no constant failure rates. In this way, an alternative is the use of a Weibull distribution which is the most commonly used distribution to model reliability data since it is easy to interpret, has great flexibility of fit and is an extension of the exponential distribution.

The probability density function (PDF) of a continuous random variable X with a Weibull distribution is given by $f_W(x;\alpha,\beta) = \alpha\beta^\alpha x^{\alpha-1}\exp\{-\beta x^\alpha\}$, where $x \geq 0$, $\beta > 0$ is the scale parameter and $\alpha > 0$ is the shape parameter. Their corresponding cumulative distribution function (CDF) and SF are given respectively by $F_W(x;\alpha,\beta) = 1-\exp\{-\beta x^\alpha\}$ and $S_W(x;\alpha,\beta) = \exp\{-\beta x^\alpha\}$. Assuming the fatal shock model previously described and considering Equation (1), it is possible to define the multivariate Marshall-Olkin Weibull (MMOW) distribution as an extension of the TMOW distribution. A comprehensive discussion about the MMOW_k model is presented by Kundu and Dey (2009) and a discussion assuming dependent right censorship is presented by Davarzani et al. (2015).

Definition 2.1. (Model formulation) Consider the transformation $Y_j = X_j^\sigma$, that is, $X_j = Y_j^{1/\sigma}$, for $j = 1,\ldots,k$; $\sigma > 0$. Let $\boldsymbol{X} = (X_1,\ldots,X_k)$ be a random vector following a MMOW distribution denoted by $\mathrm{MMOW}_k(\lambda_1,\ldots,\lambda_{k+1},\sigma)$ with multivariate SF given by

$$\begin{aligned} S(x_1,x_2,\ldots,x_k) &= \mathrm{P}(X_1 > x_1,\ldots,X_k > x_k) \\ &= \exp\{-\lambda_1 x_1^\sigma - \ldots - \lambda_k x_k^\sigma - \lambda_{k+1}\max(x_1^\sigma,\ldots,x_k^\sigma)\}. \end{aligned} \tag{2}$$

Note that if $\sigma = 1$ in Equation (2), we obtain the multivariate Marshall-Olkin exponential distribution. In this paper, we assume the special case of $k = 3$ lifetimes, that is, the TMOW distribution, assuming a 3-component system. The SF for the lifetimes X_1, X_2 and X_3 is given by

$$\begin{aligned} S(x_1,x_2,x_3) &= \mathrm{P}(X_1 > x_1, X_2 > x_2, X_3 > x_3) \\ &= \exp\{-\lambda_1 x_1^\sigma - \lambda_2 x_2^\sigma - \lambda_3 x_3^\sigma - \lambda_4\max(x_1^\sigma,x_2^\sigma,x_3^\sigma)\}, \end{aligned} \tag{3}$$

that is,

$$S(\boldsymbol{x}) = \begin{cases} S_1(\boldsymbol{x}) = \exp\{-\lambda_{14}x_1^\sigma - \lambda_2 x_2^\sigma - \lambda_3 x_3^\sigma\}, & \text{if } \; x_2 < x_3 < x_1 \;\text{ or }\; x_3 < x_2 < x_1, \\ S_2(\boldsymbol{x}) = \exp\{-\lambda_1 x_1^\sigma - \lambda_{24}x_2^\sigma - \lambda_3 x_3^\sigma\}, & \text{if } \; x_1 < x_3 < x_2 \;\text{ or }\; x_3 < x_1 < x_2, \\ S_3(\boldsymbol{x}) = \exp\{-\lambda_1 x_1^\sigma - \lambda_2 x_2^\sigma - \lambda_{34}x_3^\sigma\}, & \text{if } \; x_2 < x_1 < x_3 \;\text{ or }\; x_1 < x_2 < x_3, \\ S_4(\boldsymbol{x}) = \exp\{-\lambda_1 x_1^\sigma - (\lambda-\lambda_1)x^\sigma\}, & \text{if } \; x_1 < x_2 = x_3 = x, \\ S_5(\boldsymbol{x}) = \exp\{-\lambda_2 x_2^\sigma - (\lambda-\lambda_2)x^\sigma\}, & \text{if } \; x_2 < x_1 = x_3 = x, \\ S_6(\boldsymbol{x}) = \exp\{-\lambda_3 x_3^\sigma - (\lambda-\lambda_3)x^\sigma\}, & \text{if } \; x_3 < x_1 = x_2 = x, \\ S_7(\boldsymbol{x}) = \exp\{-\lambda x^\sigma\}, & \text{if } \; x_1 = x_2 = x_3 = x, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where $\lambda = \lambda_1+\lambda_2+\lambda_3+\lambda_4$, $\lambda_{14} = \lambda_1+\lambda_4$, $\lambda_{24} = \lambda_2+\lambda_4$ and $\lambda_{34} = \lambda_3+\lambda_4$. In addition, the PDF for the random vector $\boldsymbol{X} = (X_1,X_2,X_3)$ is obtained from $f(\boldsymbol{x}) = f(x_1,x_2,x_3) =$

$-\partial^3 S(x_1,x_2,x_3)/\partial x_1\partial x_2\partial x_3$, where $S(x_1,x_2,x_3)$ is given in Equation (4), that is,

$$f(\boldsymbol{x})=\begin{cases} f_1(\boldsymbol{x})=\lambda_{14}\lambda_2\lambda_3\sigma^3(x_1x_2x_3)^{\sigma-1}\exp\{-\lambda_{14}x_1^\sigma-\lambda_2x_2^\sigma-\lambda_3x_3^\sigma\}, & \text{if} \quad x_2<x_3<x_1 \quad \text{or} \quad x_3<x_2<x_1,\\ f_2(\boldsymbol{x})=\lambda_1\lambda_{24}\lambda_3\sigma^3(x_1x_2x_3)^{\sigma-1}\exp\{-\lambda_1x_1^\sigma-\lambda_{24}x_2^\sigma-\lambda_3x_3^\sigma\}, & \text{if} \quad x_1<x_3<x_2 \quad \text{or} \quad x_3<x_1<x_2,\\ f_3(\boldsymbol{x})=\lambda_1\lambda_2\lambda_{34}\sigma^3(x_1x_2x_3)^{\sigma-1}\exp\{-\lambda_1x_1^\sigma-\lambda_2x_2^\sigma-\lambda_{34}x_3^\sigma\}, & \text{if} \quad x_2<x_1<x_3 \quad \text{or} \quad x_1<x_2<x_3,\\ f_4(\boldsymbol{x})=\lambda_1\lambda_4\sigma^2(x_1x)^{\sigma-1}\exp\{-\lambda_1x_1^\sigma-(\lambda-\lambda_1)x^\sigma\}, & \text{if} \quad x_1<x_2=x_3=x,\\ f_5(\boldsymbol{x})=\lambda_2\lambda_4\sigma^2(x_2x)^{\sigma-1}\exp\{-\lambda_2x_2^\sigma-(\lambda-\lambda_2)x^\sigma\}, & \text{if} \quad x_2<x_1=x_3=x,\\ f_6(\boldsymbol{x})=\lambda_3\lambda_4\sigma^2(x_3x)^{\sigma-1}\exp\{-\lambda_3x_3^\sigma-(\lambda-\lambda_3)x^\sigma\}, & \text{if} \quad x_3<x_1=x_2=x,\\ f_7(\boldsymbol{x})=\lambda_4\sigma x^{\sigma-1}\exp\{-\lambda x^\sigma\}, & \text{if} \quad x_1=x_2=x_3=x,\\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

3. Classical inference for the TMOW with complete data

Let $(X_{11},X_{21},X_{31}),\ldots,(X_{1n},X_{2n},X_{3n})$ be a random sample of size n from a TMOW distribution with PDF given in (5). Consider the indicator variables stated as

$$v_1=\begin{cases}1, & \text{if} \quad x_2<x_3<x_1 \quad \text{or} \quad x_3<x_2<x_1,\\ 0, & \text{otherwise;}\end{cases}$$

$$v_2=\begin{cases}1, & \text{if} \quad x_1<x_3<x_2 \quad \text{or} \quad x_3<x_1<x_2,\\ 0, & \text{otherwise;}\end{cases}$$

$$v_3=\begin{cases}1, & \text{if} \quad x_2<x_1<x_3 \quad \text{or} \quad x_1<x_2<x_3,\\ 0, & \text{otherwise;}\end{cases}$$

$$v_4=\begin{cases}1, & \text{if} \quad x_1<x_2=x_3=x,\\ 0, & \text{otherwise;}\end{cases}$$

$$v_5=\begin{cases}1, & \text{if} \quad x_2<x_1=x_3=x,\\ 0, & \text{otherwise;}\end{cases}$$

$$v_6=\begin{cases}1, & \text{if} \quad x_3<x_1=x_2=x,\\ 0, & \text{otherwise.}\end{cases} \tag{6}$$

Then, we have seven possible situations considering the indicator variables defined as

- $r_1=v_1(1-v_2)(1-v_3)(1-v_4)(1-v_5)(1-v_6)$, if $x_2<x_3<x_1$ or $x_3<x_2<x_1$;
- $r_2=v_2(1-v_1)(1-v_3)(1-v_4)(1-v_5)(1-v_6)$, if $x_1<x_3<x_2$ or $x_3<x_1<x_2$;
- $r_3=v_3(1-v_2)(1-v_1)(1-v_4)(1-v_5)(1-v_6)$, if $x_2<x_1<x_3$ or $x_1<x_2<x_3$;
- $r_4=v_4(1-v_2)(1-v_3)(1-v_1)(1-v_5)(1-v_6)$, if $x_1<x_2=x_3=x$;
- $r_5=v_5(1-v_2)(1-v_3)(1-v_4)(1-v_1)(1-v_6)$, if $x_2<x_1=x_3=x$;
- $r_6=v_6(1-v_2)(1-v_3)(1-v_4)(1-v_5)(1-v_1)$, if $x_3<x_1=x_2=x$;
- $r_7=(1-v_1)(1-v_2)(1-v_3)(1-v_4)(1-v_5)(1-v_6)$, if $x_1=x_2=x_3=x$.

From Equation (12), the log-likelihood function assuming a TMOW distribution and a random sample of size n of lifetimes X_1, X_2 and X_3 is given by

$$\ell(\boldsymbol{\theta})=\sum_{i=1}^n r_{1i}\log\left(\lambda_{14}\lambda_2\lambda_3\sigma^3\right)+\sum_{i=1}^n r_{1i}(\sigma-1)\log(x_{1i}x_{2i}x_{3i})+\sum_{i=1}^n r_{7i}(\sigma-1)\log(x_i)$$

$$
\begin{aligned}
&+\sum_{i=1}^{n} r_{1i}\left[-\lambda_{14}x_{1i}^{\sigma}-\lambda_{2}x_{2i}^{\sigma}-\lambda_{3}x_{3i}^{\sigma}\right]+\sum_{i=1}^{n} r_{2i}\log\left(\lambda_{1}\lambda_{24}\lambda_{3}\sigma^{3}\right)+\sum_{i=1}^{n} r_{7i}\log\left(\lambda_{4}\sigma\right)\\
&+\sum_{i=1}^{n} r_{2i}(\sigma-1)\log(x_{1i}x_{2i}x_{3i})+\sum_{i=1}^{n} r_{2i}\left[-\lambda_{1}x_{1i}^{\sigma}-\lambda_{24}x_{2i}^{\sigma}-\lambda_{3}x_{3i}^{\sigma}\right]-\lambda\sum_{i=1}^{n} r_{7i}x_{i}^{\sigma}\\
&+\sum_{i=1}^{n} r_{3i}\log\left(\lambda_{1}\lambda_{2}\lambda_{34}\sigma^{3}\right)+\sum_{i=1}^{n} r_{3i}(\sigma-1)\log(x_{1i}x_{2i}x_{3i})+\sum_{i=1}^{n} r_{4i}\log\left(\lambda_{1}\lambda_{4}\sigma^{2}\right)\\
&+\sum_{i=1}^{n} r_{4i}(\sigma-1)\log(x_{1i}x_{i})+\sum_{i=1}^{n} r_{4i}\left[-\lambda_{1}x_{1i}^{\sigma}-(\lambda-\lambda_{1})x_{i}^{\sigma}\right]+\sum_{i=1}^{n} r_{5i}\log\left(\lambda_{2}\lambda_{4}\sigma^{2}\right)\\
&+\sum_{i=1}^{n} r_{5i}(\sigma-1)\log(x_{2i}x_{i})+\sum_{i=1}^{n} r_{5i}\left[-\lambda_{2}x_{2i}^{\sigma}-(\lambda-\lambda_{2})x_{i}^{\sigma}\right]+\sum_{i=1}^{n} r_{6i}\log\left(\lambda_{3}\lambda_{4}\sigma^{2}\right)\\
&+\sum_{i=1}^{n} r_{6i}(\sigma-1)\log(x_{3i}x_{i})+\sum_{i=1}^{n} r_{6i}\left[-\lambda_{3}x_{3i}^{\sigma}-(\lambda-\lambda_{3})x_{i}^{\sigma}\right]\\
&+\sum_{i=1}^{n} r_{3i}\left[-\lambda_{1}x_{1i}^{\sigma}-\lambda_{2}x_{2i}^{\sigma}-\lambda_{34}x_{3i}^{\sigma}\right]. \qquad (7)
\end{aligned}
$$

The equations for the ML estimators are presented in Appendix 1. Since the ML estimators do not have closed form, it is needed to use numerical methods as the Newton-Raphson, the Nelder-Mead or the quasi-Newton methods to get the ML estimators for each parameter of the model.

4. Classical inference for the TMOW with censored data

A particularity in the analysis of lifetime data is the presence of censored data, that could be right, left or interval censoring. In this section, we assume the presence of right censored data, that is, associated with each lifetime X_j, for $j = 1, 2, 3$, we have a fixed censoring time C_j and the data are given by $T_1 = \min(X_1, C_1)$, $T_2 = \min(X_2, C_2)$ and $T_3 = (X_3, C_3)$. In this way, the likelihood function for the parameters of the TMOW distribution has the data set classified in eight regions stated as

- B_1: T_1, T_2 and T_3 are complete observations;
- B_2: T_1 is complete, T_2 and T_3, are censored observations;
- B_3: T_1 is censored, T_2 is complete and T_3 is a censored observation;
- B_4: T_1 and T_2 are censored and T_3 is a complete observation;
- B_5: T_1 and T_2 are complete and T_3 is a censored observation;
- B_6: T_1 is complete, T_2 is censored and T_3 is a complete observation;
- B_7: T_1 is censored, T_2 and T_3 are complete observations;
- B_8: T_1, T_2 and T_3 are censored observations.

Thus, the likelihood function for $\boldsymbol{\theta} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \sigma)$ based on n observations $\boldsymbol{t}_i = (t_{1i}, t_{2i}, t_{3i})$, for $i = 1, \ldots, n$, is given by

$$
\begin{aligned}
L(\boldsymbol{\theta}) = &\prod_{i\in B_1} f(\boldsymbol{t}_i)\prod_{i\in B_2}\left(-\frac{\partial S(\boldsymbol{t}_i)}{\partial \boldsymbol{t}_{1i}}\right)\prod_{i\in B_3}\left(-\frac{\partial S(\boldsymbol{t}_i)}{\partial \boldsymbol{t}_{2i}}\right)\prod_{i\in B_4}\left(-\frac{\partial S(\boldsymbol{t}_{3i})}{\partial \boldsymbol{t}_{3i}}\right)\prod_{i\in B_5}\left(\frac{\partial^2 S(\boldsymbol{t}_i)}{\partial \boldsymbol{t}_{1i}\partial \boldsymbol{t}_{2i}}\right)\\
&\times\prod_{i\in B_6}\left(\frac{\partial^2 S(\boldsymbol{t}_i)}{\partial \boldsymbol{t}_{1i}\partial \boldsymbol{t}_{3i}}\right)\prod_{i\in B_7}\left(\frac{\partial^2 S(\boldsymbol{t}_i)}{\partial \boldsymbol{t}_{2i}\partial \boldsymbol{t}_{3i}}\right)\prod_{i\in B_8} S(\boldsymbol{t}_i), \qquad (8)
\end{aligned}
$$

where $S(\boldsymbol{t})$ is defined by Equations (3) and (4). Define the indicator variables for the censored data as

$$\delta_{ji} = \begin{cases} 1, & \text{if} \quad T_{ji} \leq C_{ji}, \\ 0, & \text{if} \quad T_{ji} > C_{ji}, \end{cases} \tag{9}$$

with $j = 1, 2, 3$ and $i = 1, \ldots, n$. In this way, the logarithm of the likelihood function stated in Equation (8) using the results $(a), (b), \ldots, (h)$ presented in Appendix 1 for the TMOW distribution in presence of right censored data is given by

$$\begin{aligned}
\ell(\boldsymbol{\theta}) = & \sum_{i=1}^{n} \delta_{1i}\delta_{2i}\delta_{3i}r_{1i} \log f_1(\boldsymbol{t}_i) + \sum_{i=1}^{n} \delta_{1i}\delta_{2i}\delta_{3i}r_{2i} \log f_2(\boldsymbol{t}_i) + \sum_{i=1}^{n} \delta_{1i}\delta_{2i}\delta_{3i}r_{3i} \log f_3(\boldsymbol{t}_i) \\
& + \sum_{i=1}^{n} \delta_{1i}\delta_{2i}\delta_{3i}r_{4i} f_4(\boldsymbol{t}_i) + \sum_{i=1}^{n} \delta_{1i}\delta_{2i}\delta_{3i}r_{5i} \log f_5(\boldsymbol{t}_i) + \sum_{i=1}^{n} \delta_{1i}\delta_{2i}\delta_{3i}r_{6i} \log f_6(\boldsymbol{t}_i) \\
& + \sum_{i=1}^{n} \delta_{1i}\delta_{2i}\delta_{3i}r_{7i} \log f_7(\boldsymbol{t}_i) + \sum_{i=1}^{n} \delta_{1i}(1-\delta_{2i})(1-\delta_{3i})r_{1i} \log g_{11}(\boldsymbol{t}_i) \\
& + \sum_{i=1}^{n} \delta_{1i}(1-\delta_{2i})(1-\delta_{3i})r_{2i} \log g_{12}(\boldsymbol{t}_i) + \sum_{i=1}^{n} \delta_{1i}(1-\delta_{2i})(1-\delta_{3i})r_{3i} \log g_{13}(\boldsymbol{t}_i) \\
& + \sum_{i=1}^{n} \delta_{1i}(1-\delta_{2i})(1-\delta_{3i})r_{4i} \log g_{14}(\boldsymbol{t}_i) + \sum_{i=1}^{n} \delta_{1i}(1-\delta_{2i})(1-\delta_{3i})r_{5i} \log g_{15}(\boldsymbol{t}_i) \\
& + \sum_{i=1}^{n} \delta_{1i}(1-\delta_{2i})(1-\delta_{3i})r_{7i} \log g_{17}(\boldsymbol{t}_i) + \sum_{i=1}^{n} (1-\delta_{1i})\delta_{2i}(1-\delta_{3i})r_{1i} \log g_{21}(\boldsymbol{t}_i) \\
& + \sum_{i=1}^{n} (1-\delta_{1i})\delta_{2i}(1-\delta_{3i})r_{3i} \log g_{23}(\boldsymbol{t}_i) + \sum_{i=1}^{n} (1-\delta_{1i})\delta_{2i}(1-\delta_{3i})r_{4i} \log g_{24}(\boldsymbol{t}_i) \\
& + \sum_{i=1}^{n} (1-\delta_{1i})(1-\delta_{2i})\delta_{3i}r_{2i} \log g_{32}(\boldsymbol{t}_i) + \sum_{i=1}^{n} (1-\delta_{1i})(1-\delta_{2i})\delta_{3i}r_{3i} \log g_{33}(\boldsymbol{t}_i) \\
& + \sum_{i=1}^{n} (1-\delta_{1i})(1-\delta_{2i})(1-\delta_{3i})r_{6i} \log S_5(\boldsymbol{t}_i) + \sum_{i=1}^{n} \delta_{1i}\delta_{2i}(1-\delta_{3i})r_{3i} \log g_{43}(\boldsymbol{t}_i) \\
& + \sum_{i=1}^{n} (1-\delta_{1i})(1-\delta_{2i})(1-\delta_{3i})r_{3i} \log S_3(\boldsymbol{t}_i) + \sum_{i=1}^{n} \delta_{1i}\delta_{2i}(1-\delta_{3i})r_{2i} \log g_{42}(\boldsymbol{t}_i) \\
& + \sum_{i=1}^{n} \delta_{1i}\delta_{2i}(1-\delta_{3i})r_{4i} \log g_{44}(\boldsymbol{t}_i) + \sum_{i=1}^{n} \delta_{1i}\delta_{2i}(1-\delta_{3i})r_{5i} \log g_{45}(\boldsymbol{t}_i) \\
& + \sum_{i=1}^{n} \delta_{1i}\delta_{2i}(1-\delta_{3i})r_{7i} \log g_{47}(\boldsymbol{t}_i) + \sum_{i=1}^{n} \delta_{1i}(1-\delta_{2i})\delta_{3i}r_{3i} \log g_{53}(\boldsymbol{t}_i) \\
& + \sum_{i=1}^{n} (1-\delta_{1i})(1-\delta_{2i})(1-\delta_{3i})r_{7i} \log S_7(\boldsymbol{t}_i) + \sum_{i=1}^{n} \delta_{1i}(1-\delta_{2i})\delta_{3i}r_{2i} \log g_{52}(\boldsymbol{t}_i)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=1}^{n} \delta_{1i}(1-\delta_{2i})\delta_{3i} r_{6i} \log g_{56}(\boldsymbol{t}_i) + \sum_{i=1}^{n}(1-\delta_{1i})\delta_{2i}\delta_{3i} r_{1i} \log g_{61}(\boldsymbol{t}_i) \\
&+ \sum_{i=1}^{n} \delta_{1i}(1-\delta_{2i})\delta_{3i} r_{1i} \log g_{51}(\boldsymbol{t}_i) + \sum_{i=1}^{n}(1-\delta_{1i})(1-\delta_{2i})(1-\delta_{3i}) r_{2i} \log S_2(\boldsymbol{t}_i) \\
&+ \sum_{i=1}^{n}(1-\delta_{1i})\delta_{2i}\delta_{3i} r_{3i} \log g_{63}(\boldsymbol{t}_i) + \sum_{i=1}^{n}(1-\delta_{1i})(1-\delta_{2i})(1-\delta_{3i}) r_{1i} \log S_1(\boldsymbol{t}_i) \\
&+ \sum_{i=1}^{n} \delta_{1i}\delta_{2i}(1-\delta_{3i}) r_{1i} \log g_{41}(\boldsymbol{t}_i) + \sum_{i=1}^{n}(1-\delta_{1i})(1-\delta_{2i})(1-\delta_{3i}) r_{4i} \log S_4(\boldsymbol{t}_i) \\
&+ \sum_{i=1}^{n}(1-\delta_{1i})(1-\delta_{2i})\delta_{3i} r_{6i} \log g_{36}(\boldsymbol{t}_i) + \sum_{i=1}^{n}(1-\delta_{1i})(1-\delta_{2i})(1-\delta_{3i}) r_{6i} \log S_6(\boldsymbol{t}_i) \\
&+ \sum_{i=1}^{n}(1-\delta_{1i})\delta_{2i}\delta_{3i} r_{2i} \log g_{62}(\boldsymbol{t}_i) + \sum_{i=1}^{n}(1-\delta_{1i})\delta_{2i}(1-\delta_{3i}) r_{2i} \log g_{22}(\boldsymbol{t}_i) \\
&+ \sum_{i=1}^{n} \delta_{1i}(1-\delta_{2i})(1-\delta_{3i}) r_{6i} \log g_{16}(\boldsymbol{t}_i) + \sum_{i=1}^{n}(1-\delta_{1i})\delta_{2i}(1-\delta_{3i}) r_{5i} \log g_{25}(\boldsymbol{t}_i).
\end{aligned}
$$

5. Simulation study

This section reports the results of a MC simulation study carried out to assess the performance of the ML estimators of the TMOW distribution assuming complete data.

The computations for classical approach were performed using `maxLik` package (Henningsen and Toomet, 2011) from the `R` software (R Core Team, 2015) with the option `optim.method = ‘‘BFGS’’` for `maxLik` function. To apply the proposed Bayesian approach, we have considered the Gibbs Sampling algorithm available in the package `R2jags` (Su and Yajima, 2012) from the `R` and `JAGS` software. A chain with $N = 100,000$ values was generated for each parameter, considering a burn-in of 5% of the size of the chain. In addition, a value generated for every 100 was considered, resulting in chains of size $1,000$ for each parameter. Furthermore, using trace plots and Geweke's diagnostic, the convergence of the chains was monitored, and their stationarity was revealed. Computer codes are available under request.

To estimate the parameters of the TMOW distribution, based on the squared error loss function, $L(\eta, a) = (\eta - a)^2$, we consider that the joint posterior PDF of the parameter $\boldsymbol{\Phi} = (\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4, \sigma)$ is obtained directly from the Bayes formula assuming independent non-informative gamma prior distributions with hyperparameters equals to $\alpha = 0.0001$ and $\beta = 0.0001$ for each parameter and is written as $\pi(\boldsymbol{\theta}; \text{data}) = L(\boldsymbol{\Phi})\pi(\sigma)\prod \pi(\lambda_i) / \int L(\boldsymbol{\theta})\pi(\sigma)\prod \pi(\lambda_i)\, \mathrm{d}\sigma \mathrm{d}\theta_i$, for $= 1, 2, 3, 4$.

The generation of the random values X_1, X_2 and X_3 from the TMOW distribution follows the steps: (1) generate $U_1 \sim \text{Weibull}(\lambda_1, \sigma)$, $U_2 \sim \text{Weibull}(\lambda_2, \sigma)$, $U_3 \sim \text{Weibull}(\lambda_3, \sigma)$, $U_4 \sim \text{Weibull}(\lambda_4, \sigma)$; (2) define $X_1 = \min(U_1, U_4), X_2 = \min(U_2, U_4)$ and $X_3 = \min(U_3, U_4)$; and (3) return the observed values (x_1, x_2, x_3) of (X_1, X_2, X_3).

The simulation study was performed under five scenarios and reported in Table 1, assuming the sample sizes equal to $n = 10, 20, 30, \ldots, 100$. In addition, it was considered 1000 MC replications for each scenario from which were computed the biases and the root of mean squared error (RMSE) as given in Equation (10). Specifically, the bias and RMSE were calculated using the expressions given by

$$\mathrm{Bias}(\widehat{\boldsymbol{\Psi}}) = \frac{1}{B}\sum_{i=1}^{B}(\widehat{\boldsymbol{\Psi}}_i - \boldsymbol{\Psi}_i), \quad \mathrm{RMSE}(\widehat{\boldsymbol{\Psi}}) = \left(\frac{1}{B}\sum_{i=1}^{B}(\widehat{\boldsymbol{\Psi}}_i - \boldsymbol{\Psi}_i)^2\right)^{1/2}, \tag{10}$$

where $B = 1000$ is the number of simulations and $\boldsymbol{\Psi}$ denotes each parameter $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ or σ. The obtained results are presented in Tables 2 and 3 from where note that:

- The biases and RMSE for parameters λ_1, λ_2 and λ_3 are high and decrease slowly to zero when $n \to \infty$ when compared to the others parameters, but, in general, the average biases and RMSE decrease when $n \to \infty$ that show the consistency property of the ML estimators. That is, we have $\mathrm{E}(\lambda_i) \approx \lambda_i, i = 1, 2, 3, 4$ and $\mathrm{E}(\sigma) \approx \sigma$ when $n \to \infty$.
- In the scenarios 3, 4 and 5, the biases for λ_4 are negatives and close to zero. The same happens to σ in scenario 4. However, for the others parameters and scenarios, the biases are positives for $\lambda_i, i = 1, 2, 3$.
- The results presented in scenario 3 has the higher values for the the biases and RMSE for $\lambda_i, i = 1, 2, 3$ and σ. For λ_4, this occur in the scenario 2. In contrast, the smaller values for the the biases and RMSE are presented in scenarios 1 and 4.
- It is important to point out that the simulation also could be made using a Bayesian approach with different prior distributions for the parameters of the TMOW distribution. The coverage probability and the coverage length could be also computed;
- The simulation results could be improved considering other random variable generation methods and using a better approach for the correlation structures of X_1, X_2 and X_3. Moreover, we conclude that the TMOW distribution could be used as a good alternative model to describe trivariate lifetimes with good accuracy in applications.

As a numeric experiment, let us consider a complete simulated data set and a censored data set (cut point equal to 2.5 for censored lifetimes) that consists of $n = 50$ trivariate lifetimes generated from the TMOW distribution assuming the parameter values presented in scenario 4 (see Table 1) for illustrative purposes of the model performance. The data sets are presented in Table 4. The inference results of interest were obtained using the `maxLik` package of the `R` software with `optim.method = 'SANN'` and are presented in Table 5 as well the asymptotic 95% confidence intervals (CIs) which were obtained using the asymptotic normal distribution given by $\mathrm{N}_5(\boldsymbol{\theta}, \boldsymbol{\Sigma}^{-1})$.

From the results presented in Table 5, we conclude that the TMOW model has a good accuracy for both simulated data sets due to the small values (< 0.5) for the standard error (SE) and the small length of the CI for each parameter which is expected since the data set was generated from the TMOW distribution.

Table 1. True parameters values for each scenario.

Scenario	λ_1	λ_2	λ_3	λ_4	σ
1	1.20	1.30	1.30	0.18	0.10
2	1.20	1.30	1.50	1.45	1.20
3	0.40	0.50	0.60	1.50	2.00
4	0.40	0.50	0.60	0.70	0.80
5	0.80	0.90	0.70	0.35	0.20

Table 2. Bias for each parameter for the considered scenarios.

	Sample Size	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5
Bias(λ_1)	10	0.9406	1.8494	2.0303	0.8178	1.2065
	20	0.7687	1.5009	1.8726	0.7527	1.0163
	30	0.7132	1.4897	1.7802	0.7260	0.9889
	40	0.6948	1.4287	1.7758	0.7225	0.9519
	50	0.6727	1.3632	1.7367	0.7131	0.9457
	60	0.6712	1.3575	1.7278	0.7090	0.9405
	70	0.6592	1.3499	1.7270	0.7059	0.9365
	80	0.6561	1.3471	1.7236	0.7053	0.9248
	90	0.6380	1.3221	1.7235	0.7041	0.9110
	100	0.6323	1.3108	1.7017	0.7006	0.9042
Bias(λ_2)	10	0.8385	2.0096	2.1121	0.8607	1.1388
	20	0.6767	1.6119	1.8778	0.7709	0.9247
	30	0.6528	1.5602	1.7946	0.7562	0.9112
	40	0.5746	1.4537	1.7654	0.7217	0.8730
	50	0.5722	1.4269	1.7236	0.7126	0.8663
	60	0.5653	1.4038	1.7084	0.7037	0.8543
	70	0.5563	1.3780	1.6938	0.7005	0.8493
	80	0.5389	1.3703	1.6889	0.6929	0.8414
	90	0.5226	1.3242	1.6825	0.6887	0.8145
	100	0.5139	1.3121	1.6395	0.6701	0.8067
Bias(λ_3)	10	0.8184	2.1178	2.4012	0.8813	1.2464
	20	0.6996	1.7690	1.9862	0.7785	1.0996
	30	0.6114	1.6260	1.8556	0.7233	1.0481
	40	0.5824	1.5543	1.8552	0.7165	1.0096
	50	0.5809	1.5376	1.7908	0.7095	1.0092
	60	0.5749	1.5191	1.7693	0.7092	1.0052
	70	0.5709	1.5141	1.7593	0.7087	1.0040
	80	0.5691	1.4873	1.7591	0.7072	1.0006
	90	0.5524	1.4720	1.7536	0.6985	0.9865
	100	0.5509	1.4612	1.7181	0.6846	0.9768
Bias(λ_4)	10	0.0248	0.0454	-0.0180	-0.0179	-0.0462
	20	0.0126	0.0378	-0.0170	-0.0169	-0.0458
	30	0.0106	0.0360	-0.0157	-0.0157	-0.0454
	40	0.0089	0.0344	-0.0145	-0.0147	-0.0451
	50	0.0085	0.0339	-0.0132	-0.0135	-0.0447
	60	0.0071	0.0338	-0.0130	-0.0123	-0.0446
	70	0.0069	0.0336	-0.0112	-0.0119	-0.0445
	80	0.0069	0.0332	-0.0104	-0.0104	-0.0432
	90	0.0064	0.0329	-0.0092	-0.0093	-0.0403
	100	0.0057	0.0326	-0.0080	-0.0073	-0.0335
Bias(σ)	10	0.0055	0.0855	0.1077	-0.0571	0.0097
	20	0.0028	0.0591	0.0666	-0.0553	0.0041
	30	0.0026	0.0519	0.0340	-0.0526	0.0031
	40	0.0021	0.0483	0.0247	-0.0523	0.0025
	50	0.0010	0.0300	0.0205	-0.0522	0.0023
	60	0.0008	0.0264	0.0097	-0.0500	0.0019
	70	0.0004	0.0244	0.0070	-0.0408	0.0016
	80	0.0002	0.0228	0.0040	-0.0372	0.0010
	90	0.0002	0.0215	0.0037	-0.0330	0.0005
	100	0.0001	0.0178	0.0014	-0.0228	0.0003

Table 3. RMSE for each parameter for the considered scenarios.

	Sample Size	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5
RMSE(λ_1)	10	1.4104	2.5790	2.3998	1.0009	1.5558
	20	0.9728	1.7555	2.0090	0.8267	1.1534
	30	0.8109	1.6609	1.8660	0.7745	1.0626
	40	0.7820	1.5619	1.8437	0.7601	1.0087
	50	0.7421	1.4531	1.7772	0.7316	0.9897
	60	0.7308	1.4406	1.7664	0.7314	0.9711
	70	0.7120	1.4122	1.7587	0.7269	0.9612
	80	0.6875	1.3879	1.7580	0.7258	0.9581
	90	0.6843	1.3820	1.7502	0.7208	0.9409
	100	0.6828	1.3725	1.7370	0.7165	0.9382
RMSE(λ_2)	10	1.1970	2.5686	2.4845	1.0251	1.4077
	20	0.8903	1.8512	2.0086	0.8477	1.0475
	30	0.7808	1.7373	1.8814	0.8081	0.9955
	40	0.6412	1.5749	1.8152	0.7424	0.9172
	50	0.6402	1.5241	1.7751	0.7403	0.9114
	60	0.6385	1.4822	1.7545	0.7391	0.9037
	70	0.6313	1.4668	1.7356	0.7286	0.8941
	80	0.5879	1.4361	1.7329	0.7119	0.8775
	90	0.5623	1.3677	1.7116	0.7020	0.8407
	100	0.5580	1.3556	1.6671	0.6855	0.8307
RMSE(λ_3)	10	1.1615	2.8146	3.0221	1.1047	1.5354
	20	0.9039	2.0753	2.1512	0.8608	1.2066
	30	0.7067	1.7595	1.9633	0.7638	1.0943
	40	0.7057	1.7371	1.9256	0.7566	1.0678
	50	0.6599	1.6315	1.8414	0.7444	1.0415
	60	0.6471	1.6293	1.8226	0.7403	1.0409
	70	0.6247	1.5752	1.7931	0.7287	1.0327
	80	0.6124	1.5382	1.7873	0.7271	1.0206
	90	0.6013	1.5333	1.7858	0.7196	1.0087
	100	0.5989	1.5181	1.7425	0.7021	0.9973
RMSE(λ_4)	10	0.0335	0.0476	0.0181	0.0180	0.0463
	20	0.0170	0.0386	0.0170	0.0170	0.0360
	30	0.0140	0.0363	0.0168	0.0168	0.0236
	40	0.0118	0.0346	0.0157	0.0158	0.0189
	50	0.0108	0.0341	0.0144	0.0146	0.0153
	60	0.0094	0.0340	0.0132	0.0135	0.0128
	70	0.0092	0.0337	0.0125	0.0121	0.0107
	80	0.0089	0.0333	0.0119	0.0117	0.0079
	90	0.0083	0.0329	0.0100	0.0108	0.0072
	100	0.0073	0.0327	0.0098	0.0100	0.0057
RMSE(σ)	10	0.0178	0.2115	0.3536	0.1174	0.0328
	20	0.0122	0.1541	0.2546	0.0908	0.0230
	30	0.0099	0.1318	0.1998	0.0800	0.0187
	40	0.0080	0.1044	0.1657	0.0799	0.0162
	50	0.0075	0.1022	0.1485	0.0723	0.0150
	60	0.0071	0.0871	0.1362	0.0692	0.0135
	70	0.0063	0.0779	0.1241	0.0665	0.0120
	80	0.0056	0.0707	0.1216	0.0661	0.0112
	90	0.0055	0.0690	0.1066	0.0655	0.0108
	100	0.0046	0.0617	0.1044	0.0639	0.0104

Table 4. Simulated data sets assuming the true parameters presented in scenario 4 for TMOW distribution.

Complete data		2.9050	1.6158	0.0360	1.4561	0.3848	1.6423	1.7376	2.4365	0.2093	0.8411	1.4982	0.3554	5.4551
	X_1:	0.9944	0.8322	0.2012	0.0784	0.7711	0.0184	1.7222	0.6561	2.3231	2.2312	2.8298	0.1304	1.8389
		0.0566	0.7413	0.5567	0.2910	3.7102	1.1128	2.5874	0.2352	0.2228	1.1922	0.8506	1.1641	0.1689
		1.9354	1.3771	0.8613	0.7622	2.5632	0.0054	1.0581	1.3065	0.3640	0.0045	1.8680		
		2.9050	1.3486	0.8815	0.3376	0.3848	0.7980	1.7376	0.1366	0.2093	0.8411	2.1215	0.4391	2.3127
	X_2:	1.0389	0.6509	0.2012	0.0784	0.7711	2.5113	1.7222	0.1462	0.0569	1.9140	1.2382	1.5333	0.0373
		0.3128	2.0642	0.1847	0.2930	0.5026	1.1128	1.4667	0.0727	0.2228	0.0352	0.8506	2.5642	0.1689
		2.3867	1.3771	1.0442	0.7622	2.7941	0.0054	0.3199	0.5004	1.5470	1.4413	2.9805		
		1.5124	1.4561	2.5641	0.5822	0.3848	0.3794	1.7376	0.5608	0.2093	0.1380	0.4593	0.9403	0.0971
	X_3:	1.4204	0.4511	0.2012	0.0784	0.7711	1.7915	1.7222	0.6561	0.7871	0.4913	2.8522	0.2301	3.5971
		0.3128	0.7610	0.5567	0.2930	1.4540	0.0052	2.0376	0.7102	0.0016	2.8227	0.8506	0.3387	0.1689
		3.4390	0.0107	0.6489	0.7622	1.9607	0.0054	1.2430	1.3065	1.5470	1.2935	0.9869		
Censored data		2.9050	1.6158	0.0360	1.4561	0.3848	1.6423	1.7376	2.4365	0.2093	0.8411	1.4982	0.3554	5.4551
	X_1:	0.9944	0.8322	0.2012	0.0784	0.7711	0.0184	1.7222	0.6561	2.3231	2.2312	2.8298	0.1304	1.8389
		0.0566	0.7413	0.5567	0.2910	3.7102	1.1128	2.5874	0.2352	0.2228	1.1922	0.8506	1.1641	0.1689
		1.9354	1.3771	0.8613	0.7622	2.5632	0.0054	1.0581	1.3065	0.3640	0.0045	1.8680		
		2.9050	1.3486	0.8815	0.3376	0.3848	0.7980	1.7376	0.1366	0.2093	0.8411	2.1215	0.4391	2.3127
	X_2:	1.0389	0.6509	0.2012	0.0784	0.7711	2.5113	1.7222	0.1462	0.0569	1.9140	1.2382	1.5333	0.0373
		0.3128	2.0642	0.1847	0.2930	0.5026	1.1128	1.4667	0.0727	0.2228	0.0352	0.8506	2.5642	0.1689
		2.3867	1.3771	1.0442	0.7622	2.7941	0.0054	0.3199	0.5004	1.5470	1.4413	2.9805		
		1.5124	1.4561	2.5641	0.5822	0.3848	0.3794	1.7376	0.5608	0.2093	0.1380	0.4593	0.9403	0.0971
	X_3:	1.4204	0.4511	0.2012	0.0784	0.7711	1.7915	1.7222	0.6561	0.7871	0.4913	2.8522	0.2301	3.5971
		0.3128	0.7610	0.5567	0.2930	1.4540	0.0052	2.0376	0.7102	0.0016	2.8227	0.8506	0.3387	0.1689
		3.4390	0.0107	0.6489	0.7622	1.9607	0.0054	1.2430	1.3065	1.5470	1.2935	0.9869		

Table 5. ML estimates and the corresponding SE for the model parameters (both simulated data sets).

Parameter	Complete data			Censored data		
	ML	SE	95% CI	ML	SE	95% CI
λ_1	0.4276	0.0915	(0.2482, 0.6069)	0.3831	0.0847	(0.2172, 0.5491)
λ_2	0.5623	0.1164	(0.3342, 0.7904)	0.5705	0.1030	(0.3687, 0.7723)
λ_3	0.6489	0.1282	(0.3976, 0.9001)	0.4703	0.0970	(0.2801, 0.6604)
λ_4	0.7489	0.1441	(0.4664, 1.0314)	0.7629	0.1366	(0.4951, 1.0306)
σ	0.8109	0.0660	(0.6815, 0.9402)	0.7792	0.0633	(0.6553, 0.9032)

6. Application to real reliability data

To illustrate the proposed model, let us assume a reliability data set introduced by Crowder et al. (1994). This data set consists of fiber failure strengths. The four values in each row give the breaking strengths of fiber sections of lengths 5, 12, 30 and 75mm. The values are right-censored at 4.0 and a zero indicates accidental breakage prior to testing; the zeros have been treated as missing data. The data sets are available in Table 7.2 from Crowder et al. (1994). In view of the apparent heterogeneity between fibers a model allowing individual random levels would be appropriate and the proposed TMOW model could be useful in the data analysis. In this way, we assume as response lifetimes the length equals 12mm as X_1, the length equals 30mm as X_2 and the length equals 75mm as X_3. Firstly, to apply the proposed methodology under a right-censored scheme, we have considered the Classical approach. The inference results of interest were obtained using the `maxLik` package of the `R` software with the option `optim.method = ''BFGS''` for `maxLik` function and are presented in Table 6 as well the asymptotic 95% CIs which were obtained using the asymptotic normal distribution given by $\mathrm{N}_5(\boldsymbol{\theta}, \boldsymbol{\Sigma}^{-1})$.

Table 6. ML estimates for fiber failure strengths data sets.

Parameter	Data set 1			Data set 2			Data set 3		
	ML	SE	95% CI	ML	SE	95% CI	ML	SE	95% CI
λ_1	0.00008	0.00008	(-0.00008, 0.00024)	0.00019	5.93164	(-11.6256, 11.6260)	0.00013	0.00013	(-0.00012, 0.00038)
λ_2	0.00095	0.00071	(-0.00044, 0.00234)	0.00845	0.00131	(0.00588, 0.01102)	0.00192	0.00119	(-0.00041, 0.00425)
λ_3	0.00353	0.00209	(-0.00057, 0.00763)	0.02057	0.00331	(0.01408, 0.02706)	0.00420	0.00245	(-0.00061, 0.00910)
λ_4	0.00002	0.00003	(-0.00004, 0.00008)	0.00108	5.93164	(-11.6247, 11.6269)	0.00001	0.00017	(-0.00033, 0.00033)
σ	7.08472	0.69575	(5.72108, 8.44836)	6.58032	0.47135	(5.65649, 7.50415)	6.86184	0.63028	(5.62652, 8.09716)

From the results displayed in Table 6, one can notice that there is an instability using the classical approach (negative bounds for 95% CI, high values for standard errors), especially for Data set 2. This fact may be related to the complexity of the likelihood in presence of right-censored data. Thus, to avoid this problem, a Bayesian method was considered (see Appendix 1). The inference results of interest for each data set are presented in Table 7 and the plots of the marginal posterior densities for the parameters of the model considering each data set are presented in Figure 1.

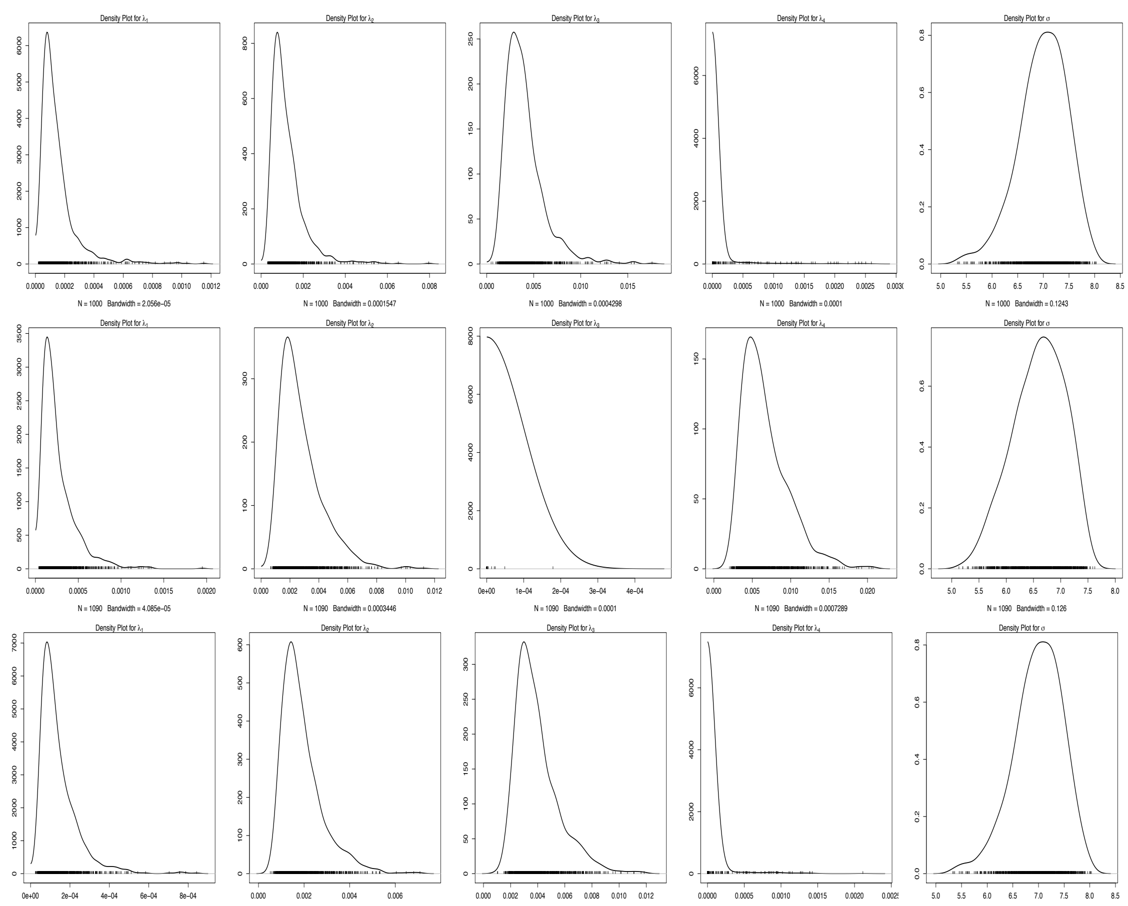

Figure 1. Posterior PDF plots for the parameters of the model assuming the three failure strength data sets (top: Data set 1; middle: Data set 2; bottom: Data set 3).

Table 7. Bayesian estimate, credible interval (CrI) and corresponding standard deviation (SD) for fiber failure strengths data sets.

Parameter	Data set 1			Data set 2			Data set 3		
	Mean	SD	95% CrI	Mean	SD	95% CrI	Mean	SD	95% CrI
λ_1	0.00015	0.00013	(0.00003, 0.00052)	0.00026	0.00017	(0.00006, 0.00064)	0.00014	0.00010	(0.00004, 0.00041)
λ_2	0.00123	0.00081	(0.00040, 0.00333)	0.00273	0.00135	(0.00104, 0.00581)	0.00191	0.00091	(0.00082, 0.00426)
λ_3	0.00399	0.00211	(0.00147, 0.00936)	0.00002	0.00001	(0.00001, 0.00003)	0.00395	0.00161	(0.00190, 0.00800)
λ_4	0.00013	0.00035	(0.00001, 0.00120)	0.00661	0.00287	(0.00263, 0.01291)	0.00001	0.00007	(921E-08, 0.00015)
σ	6.98866	0.47595	(5.92113, 7.79890)	6.57077	0.48138	(5.68461, 7.39571)	6.93945	0.39701	(6.08930, 7.53128)

From the results obtained, note that, for Data sets 1 and 3, the estimate of the parameter λ_4 is very close to zero that means its contribution for the likelihood function is very small. The same happens to the parameter λ_3 in Data set 2. In general, we conclude that the posterior SD values approach to zero and the 95% CrI have reasonable lengths.

7. Concluding remarks

In this paper, we introduced a new trivariate distribution obtained as a special case of the multivariate Marshall-Olkin Weibull distribution. For this new model, we presented some inference properties and an extensive simulation study was performed to verify the performance of the maximum likelihood estimators assuming different fixed values for the parameters of the model and different sample sizes.

The obtained results from Monte Carlo studies showed that the bias and root of mean squared error of the estimators of the trivariate Marshall-Olkin-Weibull distribution are asymptotically non-biased and approaches to zero when the sample size increases even assuming negative values for the biases in some scenarios. From these results, it is possible to conclude that using the proposed model, the obtained inference results are reasonable accurate considering complete data sets and with good performance of the computational algorithm used to get the inferences of interest. However, in the application of fiber strengths, there was a problem with maximum likelihood estimators leading to negative bound for the 95% confidence intervals which could be related of the likelihood function under a right-censoring scheme. To avoid this problem, we considered a Bayesian estimator that provide a better accuracy and good convergence of the simulation algorithm used to get the inference results of interest even using approximately non-informative prior distributions.

In conclusion, the trivariate Marshall-Olkin-Weibull distribution could be used as an alternative to model trivariate data which could be interesting for the reliability analysis (as the fiber strength application) used in engineering applications, or others areas of interest, especially considering a Bayesian approach to estimate the parameters. It is important to point out that other approaches also could be used to get inferences of the proposed model using the expectation-maximization algorithm (Kundu and Dey, 2009) but this topic will be the goal of other study.

Acknowledgments

The authors are grateful to the Editors and Reviewers for several suggestions that led to an improvement of the manuscript.

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Appendix

ML estimators with complete data

From Equations (5) and (6), the likelihood function for $\boldsymbol{\theta} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \sigma)$ assuming a TMOW distribution and a random sample of size n of the lifetimes X_1, X_2 and X_3 is given by

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n}[f_1(\boldsymbol{x}_i)]^{r_{1i}} \prod_{i=1}^{n}[f_2(\boldsymbol{x}_i)]^{r_{2i}} \prod_{i=1}^{n}[f_3(\boldsymbol{x}_i)]^{r_{3i}} \prod_{i=1}^{n}[f_4(\boldsymbol{x}_i)]^{r_{4i}} \prod_{i=1}^{n}[f_5(\boldsymbol{x}_i)]^{r_{5i}} \prod_{i=1}^{n}[f_6(\boldsymbol{x}_i)]^{r_{6i}} \times \prod_{i=1}^{n}[f_7(\boldsymbol{x}_i)]^{r_{7i}}. \tag{11}$$

where $\boldsymbol{x}_i = (x_{1i}, x_{2i}, x_{3i})$, for $i = 1, \ldots, n$; $f_1(\boldsymbol{x}_i)$, $f_2(\boldsymbol{x}_i)$ and $f_3(\boldsymbol{x}_i)$ are given in Equation (5). In this way, the likelihood function stated in Equation (11) can be rewritten as

$$\begin{aligned}
L(\boldsymbol{\theta}) = {} & \left(\lambda_{14}\lambda_2\lambda_3\sigma^3\right)^{\sum_{i=1}^{n} r_{1i}} \prod_{i=1}^{n}(x_{1i}x_{2i}x_{3i})^{r_{1i}(\sigma-1)} \exp\left\{ -\lambda_{14}\sum_{i=1}^{n} r_{1i}x_{1i}^{\sigma} - \lambda_2\sum_{i=1}^{n} r_{1i}x_{2i}^{\sigma} \right.\\
& \left. -\lambda_3\sum_{i=1}^{n} r_{1i}x_{3i}^{\sigma}\right\} \left(\lambda_1\lambda_{24}\lambda_3\sigma^3\right)^{\sum_{i=1}^{n} r_{2i}} \prod_{i=1}^{n}(x_{1i}x_{2i}x_{3i})^{r_{2i}(\sigma-1)} \exp\left\{ -\lambda_1\sum_{i=1}^{n} r_{2i}x_{1i}^{\sigma} \right.\\
& \left. -\lambda_{24}\sum_{i=1}^{n} r_{2i}x_{2i}^{\sigma} - \lambda_3\sum_{i=1}^{n} r_{2i}x_{3i}^{\sigma}\right\} \left(\lambda_1\lambda_2\lambda_{34}\sigma^3\right)^{\sum_{i=1}^{n} r_{3i}} \prod_{i=1}^{n}(x_{1i}x_{2i}x_{3i})^{r_{3i}(\sigma-1)}\\
& \times \exp\left\{-\lambda_1\sum_{i=1}^{n} r_{3i}x_{1i}^{\sigma} - \lambda_2\sum_{i=1}^{n} r_{3i}x_{2i}^{\sigma} - \lambda_{34}\sum_{i=1}^{n} r_{3i}x_{3i}^{\sigma}\right\} \left(\lambda_1\lambda_4\sigma^2\right)^{\sum_{i=1}^{n} r_{4i}}\\
& \times \prod_{i=1}^{n}(x_{1i}x_i)^{r_{4i}(\sigma-1)} \exp\left\{-\lambda_1\sum_{i=1}^{n} r_{4i}x_{1i}^{\sigma} - (\lambda-\lambda_1)\sum_{i=1}^{n} r_{4i}x_i^{\sigma}\right\} \left(\lambda_2\lambda_4\sigma^2\right)^{\sum_{i=1}^{n} r_{5i}}\\
& \times \prod_{i=1}^{n}(x_{2i}x_i)^{r_{5i}(\sigma-1)} \exp\left\{-\lambda_2\sum_{i=1}^{n} r_{5i}x_{2i}^{\sigma} - (\lambda-\lambda_2)\sum_{i=1}^{n} r_{5i}x_i^{\sigma}\right\} \left(\lambda_3\lambda_4\sigma^2\right)^{\sum_{i=1}^{n} r_{6i}}\\
& \times \prod_{i=1}^{n}(x_{3i}x_i)^{r_{6i}(\sigma-1)} \exp\left\{-\lambda_3\sum_{i=1}^{n} r_{6i}x_{3i}^{\sigma} - (\lambda-\lambda_3)\sum_{i=1}^{n} r_{6i}x_i^{\sigma}\right\} (\lambda_4\sigma)^{\sum_{i=1}^{n} r_{7i}}\\
& \times \prod_{i=1}^{n} x_i^{r_{7i}(\sigma-1)} \exp\left\{-\lambda\sum_{i=1}^{n} r_{7i}x_i^{\sigma}\right\}.
\end{aligned} \tag{12}$$

The ML estimators for the parameters λ_1, λ_2, λ_3, λ_4 and σ are obtained solving the equations $\partial\ell/\partial\lambda_1 = 0$, $\partial\ell/\partial\lambda_2 = 0$, $\partial\ell/\partial\lambda_3 = 0$, $\partial\ell/\partial\lambda_4 = 0$ and $\partial\ell/\partial\sigma = 0$. From the log-likelihood given in Equation (7), the first derivatives of $\ell(\boldsymbol{\theta})$ with respect to λ_1, λ_2, λ_3,

λ_4 and σ are given respectively by

$$\frac{\partial \ell}{\partial \sigma} = \frac{3}{\sigma}\sum_{i=1}^{n}[r_{1i}+r_{2i}+r_{3i}] + \frac{2}{\sigma}\sum_{i=1}^{n}[r_{4i}+r_{5i}+r_{6i}] + \frac{1}{\sigma}\sum_{i=1}^{n} r_{7i}$$
$$+\sum_{i=1}^{n}[r_{1i}+r_{2i}+r_{3i}]\log(x_{1i},x_{2i},x_{3i}) + \sum_{i=1}^{n} r_{4i}\log(x_{1i},x_i)$$
$$+\sum_{i=1}^{n} r_{5i}\log(x_{2i},x_i) + \sum_{i=1}^{n} r_{6i}\log(x_{3i},x_i)$$
$$+\sum_{i=1}^{n} r_{7i}\log(x_i) - \sum_{i=1}^{n}[\lambda_{14}r_{1i}+\lambda_1(r_{2i}+r_{3i}+r_{4i})]x_{1i}^{\sigma}\log(x_{1i})$$
$$-\sum_{i=1}^{n}[\lambda_{24}r_{2i}+\lambda_2(r_{1i}+r_{3i}+r_{5i})]x_{2i}^{\sigma}\log(x_{2i})$$
$$-\sum_{i=1}^{n}[\lambda_{34}r_{3i}+\lambda_3(r_{1i}+r_{2i}+r_{6i})]x_{1i}^{\sigma}\log(x_{1i})$$
$$-\sum_{i=1}^{n}[(\lambda-\lambda_1)r_{4i}+(\lambda-\lambda_2)r_{5i}+(\lambda-\lambda_3)r_{6i}+\lambda r_{7i}]x_i^{\sigma}\log(x_i),$$

$$\frac{\partial \ell}{\partial \lambda_1} = \frac{1}{\lambda_{14}}\sum_{i=1}^{n} r_{1i} + \frac{1}{\lambda_1}\sum_{i=1}^{n}[r_{2i}+r_{3i}+r_{4i}] - \sum_{i=1}^{n}\left[(r_{1i}+r_{2i}+r_{3i}+r_{4i})x_{1i}^{\sigma}\right]$$
$$-\sum_{i=1}^{n}\left[(r_{5i}+r_{6i}+r_{7i})x_i^{\sigma}\right],$$

$$\frac{\partial \ell}{\partial \lambda_2} = \frac{1}{\lambda_{24}}\sum_{i=1}^{n} r_{2i} + \frac{1}{\lambda_2}\sum_{i=1}^{n}[r_{1i}+r_{3i}+r_{5i}] - \sum_{i=1}^{n}\left[(r_{1i}+r_{2i}+r_{3i}+r_{5i})x_{2i}^{\sigma}\right]$$
$$-\sum_{i=1}^{n}\left[(r_{4i}+r_{6i}+r_{7i})x_i^{\sigma}\right],$$

$$\frac{\partial \ell}{\partial \lambda_3} = \frac{1}{\lambda_{34}}\sum_{i=1}^{n} r_{3i} + \frac{1}{\lambda_3}\sum_{i=1}^{n}[r_{1i}+r_{2i}+r_{6i}] - \sum_{i=1}^{n}\left[(r_{1i}+r_{2i}+r_{3i}+r_{6i})x_{3i}^{\sigma}\right]$$
$$-\sum_{i=1}^{n}\left[(r_{4i}+r_{5i}+r_{7i})x_i^{\sigma}\right],$$

$$\frac{\partial \ell}{\partial \lambda_4} = \frac{1}{\lambda_{14}}\sum_{i=1}^{n} r_{1i} + \frac{1}{\lambda_{24}}\sum_{i=1}^{n} r_{2i} + \frac{1}{\lambda_{34}}\sum_{i=1}^{n} r_{3i} + \frac{1}{\lambda_4}\sum_{i=1}^{n}[r_{4i}+r_{5i}+r_{6i}+r_{7i}]$$
$$-\sum_{i=1}^{n}\left[r_{1i}x_{1i}^{\sigma}+r_{2i}x_{2i}^{\sigma}+r_{3i}x_{3i}^{\sigma}\right] - \sum_{i=1}^{n}\left[(r_{4i}+r_{5i}+r_{6i}+r_{7i})x_i^{\sigma}\right].$$

Under standard asymptotic ML theory, confidence intervals and hypothesis tests for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and σ could be obtained from the asymptotic normality of the ML estimators $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4$ and $\hat{\sigma}$, that is, $\widehat{\boldsymbol{\theta}} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4, \hat{\sigma}) \sim \mathrm{N}_5(\boldsymbol{\theta}, \boldsymbol{\Sigma}^{-1})$, where N_5 denotes a multivariate normal distribution of dimension 5 assuming large sample sizes and $\boldsymbol{\Sigma}$ is the observed Fisher information matrix given by

$$\boldsymbol{\Sigma} = \begin{pmatrix} -\frac{\partial^2 \ell}{\partial \lambda_1^2} & -\frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_2} & -\frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_3} & -\frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_4} & -\frac{\partial^2 \ell}{\partial \lambda_1 \partial \sigma} \\ -\frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_1} & -\frac{\partial^2 \ell}{\partial \lambda_2^2} & -\frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_3} & -\frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_4} & -\frac{\partial^2 \ell}{\partial \lambda_2 \partial \sigma} \\ -\frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_1} & -\frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_2} & -\frac{\partial^2 \ell}{\partial \lambda_3^2} & -\frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_4} & -\frac{\partial^2 \ell}{\partial \lambda_3 \partial \sigma} \\ -\frac{\partial^2 \ell}{\partial \lambda_4 \partial \lambda_1} & -\frac{\partial^2 \ell}{\partial \lambda_4 \partial \lambda_2} & -\frac{\partial^2 \ell}{\partial \lambda_4 \partial \lambda_3} & -\frac{\partial^2 \ell}{\partial \lambda_4^4} & -\frac{\partial^2 \ell}{\partial \lambda_4 \partial \sigma} \\ -\frac{\partial^2 \ell}{\partial \sigma \partial \lambda_1} & -\frac{\partial^2 \ell}{\partial \sigma \partial \lambda_2} & -\frac{\partial^2 \ell}{\partial \sigma \partial \lambda_3} & -\frac{\partial^2 \ell}{\partial \sigma \partial \lambda_4} & -\frac{\partial^2 \ell}{\partial \sigma^2} \end{pmatrix}, \tag{13}$$

where all components of Equation (13) are calculated at the obtained ML estimators for the parameters of the model. The second derivatives of the log-likelihood function $\ell(\boldsymbol{\theta})$ required in the observed Fisher information matrix are given by

$$\frac{\partial^2 \ell}{\partial \lambda_1^2} = -\frac{1}{\lambda_{14}^2} \sum_{i=1}^{n} r_{1i} - \frac{1}{\lambda_1^2} \sum_{i=1}^{n} [r_{2i} + r_{3i} + r_{4i}],$$

$$\frac{\partial^2 \ell}{\partial \lambda_2^2} = -\frac{1}{\lambda_{24}^2} \sum_{i=1}^{n} r_{2i} - \frac{1}{\lambda_2^2} \sum_{i=1}^{n} [r_{1i} + r_{3i} + r_{5i}],$$

$$\frac{\partial^2 \ell}{\partial \lambda_3^2} = -\frac{1}{\lambda_{34}^2} \sum_{i=1}^{n} r_{3i} - \frac{1}{\lambda_3^2} \sum_{i=1}^{n} [r_{1i} + r_{2i} + r_{6i}],$$

$$\frac{\partial^2 \ell}{\partial \lambda_4^2} = -\frac{1}{\lambda_{14}^2} \sum_{i=1}^{n} r_{1i} - \frac{1}{\lambda_{24}^2} \sum_{i=1}^{n} r_{2i} - \frac{1}{\lambda_{34}^2} \sum_{i=1}^{n} r_{3i} - \frac{1}{\lambda_4^2} \sum_{i=1}^{n} [r_{4i} + r_{5i} + r_{6i} + r_{7i}],$$

$$\frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_4} = \frac{\partial^2 \ell}{\partial \lambda_4 \partial \lambda_1} = -\frac{1}{\lambda_{14}^2} \sum_{i=1}^{n} r_{1i},$$

$$\frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_4} = \frac{\partial^2 \ell}{\partial \lambda_4 \partial \lambda_2} = -\frac{1}{\lambda_{24}^2} \sum_{i=1}^{n} r_{2i},$$

$$\frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_4} = \frac{\partial^2 \ell}{\partial \lambda_4 \partial \lambda_3} = -\frac{1}{\lambda_{34}^2} \sum_{i=1}^{n} r_{3i},$$

$$\frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_2} = \frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_1} = \frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_3} = \frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_1} = \frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_3} = \frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_2} = 0,$$

$$\frac{\partial^2 \ell}{\partial \lambda_1 \partial \sigma} = \frac{\partial^2 \ell}{\partial \sigma \partial \lambda_1} = -\sum_{i=1}^{n} [(r_{1i} + r_{2i} + r_{3i} + r_{4i}) x_{1i}^{\sigma} \log(x_{1i})]$$

$$- \sum_{i=1}^{n} [(r_{5i} + r_{6i} + r_{7i}) x_i^{\sigma} \log(x_i)],$$

$$\frac{\partial^2 \ell}{\partial\lambda_2\partial\sigma} = \frac{\partial^2 \ell}{\partial\sigma\partial\lambda_2} = -\sum_{i=1}^{n} \left[(r_{1i}+r_{2i}+r_{3i}+r_{5i})x_{2i}^{\sigma}\log(x_{2i})\right] - \sum_{i=1}^{n}\left[(r_{4i}+r_{6i}+r_{7i})x_i^{\sigma}\log(x_i)\right],$$

$$\frac{\partial^2 \ell}{\partial\lambda_3\partial\sigma} = \frac{\partial^2 \ell}{\partial\sigma\partial\lambda_3} = -\sum_{i=1}^{n} \left[(r_{1i}+r_{2i}+r_{3i}+r_{6i})x_{2i}^{\sigma}\log(x_{3i})\right] - \sum_{i=1}^{n}\left[(r_{4i}+r_{5i}+r_{7i})x_i^{\sigma}\log(x_i)\right],$$

$$\frac{\partial^2 \ell}{\partial\lambda_4\partial\sigma} = \frac{\partial^2 \ell}{\partial\sigma\partial\lambda_4} = -\sum_{i=1}^{n} \left[r_{1i}x_{1i}^{\sigma}\log(x_{1i}) + r_{2i}x_{2i}^{\sigma}\log(x_{2i}) + r_{3i}x_{3i}^{\sigma}\log(x_{3i})\right]$$
$$-\sum_{i=1}^{n}\left[(r_{4i}+r_{5i}+r_{6i}+r_{7i})x_i^{\sigma}\log(x_i)\right],$$

$$\frac{\partial^2 \ell}{\partial\sigma^2} = -\frac{3}{\sigma^2}\sum_{i=1}^{n}[r_{1i}+r_{2i}+r_{3i}] - \frac{2}{\sigma^2}\sum_{i=1}^{n}[r_{4i}+r_{5i}+r_{6i}] - \frac{1}{\sigma^2}\sum_{i=1}^{n} r_{7i}$$
$$-\sum_{i=1}^{n}[\lambda_{14}r_{1i}+\lambda_1(r_{2i}+r_{3i}+r_{4i})]2x_{1i}^{\sigma}\log(x_{1i})$$
$$-\sum_{i=1}^{n}[\lambda_{24}r_{2i}+\lambda_2(r_{1i}+r_{3i}+r_{5i})]2x_{2i}^{\sigma}\log(x_{2i})$$
$$-\sum_{i=1}^{n}[\lambda_{34}r_{3i}+\lambda_3(r_{1i}+r_{2i}+r_{6i})]2x_{1i}^{\sigma}\log(x_{1i})$$
$$-\sum_{i=1}^{n}[(\lambda-\lambda_1)r_{4i}+(\lambda-\lambda_2)r_{5i}+(\lambda-\lambda_3)r_{6i}+\lambda r_{7i}]2x_i^{\sigma}\log(x_i).$$

Terms of the likelihood function with censored data

From Equations (9) and (6), we obtain expressions for the terms of the likelihood function defined in Equation (8) as

(a)

$$\prod\nolimits_{i\in B_1} f(\boldsymbol{t}_i) = \Big[\prod\nolimits_{i=1}^{n}[f_1(\boldsymbol{t}_i)]^{r_{1i}}\prod\nolimits_{i=1}^{n}[f_2(\boldsymbol{t}_i)]^{r_{2i}}\prod\nolimits_{i=1}^{n}[f_3(\boldsymbol{t}_i)]^{r_{3i}}\prod\nolimits_{i=1}^{n}[f_4(\boldsymbol{t}_i)]^{r_{4i}}$$
$$\times\prod\nolimits_{i=1}^{n}[f_5(\boldsymbol{t}_i)]^{r_{5i}}\prod\nolimits_{i=1}^{n}[f_6(\boldsymbol{t}_i)]^{r_{6i}}\prod\nolimits_{i=1}^{n}[f_7(\boldsymbol{t}_i)]^{r_{7i}}\Big]^{\delta_{1i}\delta_{2i}\delta_{3i}},$$

where $f_1(\boldsymbol{t}_i)$, $f_2(\boldsymbol{t}_i)$, $f_3(\boldsymbol{t}_i)$, $f_4(\boldsymbol{t}_i)$, $f_5(\boldsymbol{t}_i)$, $f_6(\boldsymbol{t}_i)$ and $f_7(\boldsymbol{t}_i)$ are defined by Equation (5).

(b)

$$\prod\nolimits_{i\in B_2}\left(-\frac{\partial S(\boldsymbol{t}_i)}{\partial \boldsymbol{t}_{1i}}\right) = \Big[\prod\nolimits_{i=1}^{n}[g_{11}(\boldsymbol{t}_i)]^{r_{1i}}\prod\nolimits_{i=1}^{n}[g_{12}(\boldsymbol{t}_i)]^{r_{2i}}\prod\nolimits_{i=1}^{n}[g_{13}(\boldsymbol{t}_i)]^{r_{3i}}\prod\nolimits_{i=1}^{n}[g_{14}(\boldsymbol{t}_i)]^{r_{4i}}$$
$$\times\prod\nolimits_{i=1}^{n}[g_{15}(\boldsymbol{t}_i)]^{r_{5i}}\prod\nolimits_{i=1}^{n}[g_{16}(\boldsymbol{t}_i)]^{r_{6i}}\prod\nolimits_{i=1}^{n}[g_{17}(\boldsymbol{t}_i)]^{r_{7i}}\Big]^{\delta_{1i}(1-\delta_{2i})(1-\delta_{3i})},$$

where

$$
\begin{aligned}
g_{11}(\boldsymbol{t}_i) &= \lambda_{14}\sigma t_{1i}^{\sigma-1}\exp\{-\lambda_{14}t_{1i}^{\sigma} - \lambda_2 t_{2i}^{\sigma} - \lambda_3 t_{3i}^{\sigma}\},\\
g_{12}(\boldsymbol{t}_i) &= \lambda_{1}\sigma t_{1i}^{\sigma-1}\exp\{-\lambda_{1}t_{1i}^{\sigma} - \lambda_{24} t_{2i}^{\sigma} - \lambda_3 t_{3i}^{\sigma}\},\\
g_{13}(\boldsymbol{t}_i) &= \lambda_{1}\sigma t_{1i}^{\sigma-1}\exp\{-\lambda_{1}t_{1i}^{\sigma} - \lambda_{2} t_{2i}^{\sigma} - \lambda_{34} t_{3i}^{\sigma}\},\\
g_{14}(\boldsymbol{t}_i) &= \lambda_{1}\sigma t_{1i}^{\sigma-1}\exp\{-\lambda_{1}t_{1i}^{\sigma} - (\lambda-\lambda_1) t_{2i}^{\sigma}\},\\
g_{15}(\boldsymbol{t}_i) &= (\lambda-\lambda_2)\sigma t_{1i}^{\sigma-1}\exp\{-\lambda_{2}t_{2i}^{\sigma} - (\lambda-\lambda_2) t_{1i}^{\sigma}\},\\
g_{16}(\boldsymbol{t}_i) &= (\lambda-\lambda_3)\sigma t_{1i}^{\sigma-1}\exp\{-\lambda_{3}t_{3i}^{\sigma} - (\lambda-\lambda_2) t_{1i}^{\sigma}\},\\
g_{17}(\boldsymbol{t}_i) &= \lambda\sigma t_{1i}^{\sigma-1}\exp\{-\lambda t_{1i}^{\sigma}\}.
\end{aligned}
$$

(c)

$$
\begin{aligned}
\prod_{i\in B_3}\left(-\frac{\partial S(\boldsymbol{t}_i)}{\partial \boldsymbol{t}_{2i}}\right) = \Bigg[&\prod_{i=1}^n [g_{21}(\boldsymbol{t}_i)]^{r_{1i}}\prod_{i=1}^n [g_{22}(\boldsymbol{t}_i)]^{r_{2i}}\prod_{i=1}^n [g_{23}(\boldsymbol{t}_i)]^{r_{3i}}\prod_{i=1}^n [g_{24}(\boldsymbol{t}_i)]^{r_{4i}}\\
&\times\prod_{i=1}^n [g_{25}(\boldsymbol{t}_i)]^{r_{5i}}\prod_{i=1}^n [g_{26}(\boldsymbol{t}_i)]^{r_{6i}}\prod_{i=1}^n [g_{27}(\boldsymbol{t}_i)]^{r_{7i}}\Bigg]^{(1-\delta_{1i})\delta_{2i}(1-\delta_{3i})},
\end{aligned}
$$

where

$$
\begin{aligned}
g_{21}(\boldsymbol{t}_i) &= \lambda_{2}\sigma t_{2i}^{\sigma-1}\exp\{-\lambda_{14}t_{1i}^{\sigma} - \lambda_2 t_{2i}^{\sigma} - \lambda_3 t_{3i}^{\sigma}\},\\
g_{22}(\boldsymbol{t}_i) &= \lambda_{24}\sigma t_{2i}^{\sigma-1}\exp\{-\lambda_{1}t_{1i}^{\sigma} - \lambda_{24} t_{2i}^{\sigma} - \lambda_3 t_{3i}^{\sigma}\},\\
g_{23}(\boldsymbol{t}_i) &= \lambda_{2}\sigma t_{2i}^{\sigma-1}\exp\{-\lambda_{1}t_{1i}^{\sigma} - \lambda_{2} t_{2i}^{\sigma} - \lambda_{34} t_{3i}^{\sigma}\},\\
g_{24}(\boldsymbol{t}_i) &= (\lambda-\lambda_1)\sigma t_{2i}^{\sigma-1}\exp\{-\lambda_{1}t_{1i}^{\sigma} - (\lambda-\lambda_1) t_{i}^{\sigma}\},\\
g_{25}(\boldsymbol{t}_i) &= \lambda_2\sigma t_{2i}^{\sigma-1}\exp\{-\lambda_{2}t_{2i}^{\sigma} - (\lambda-\lambda_2) t_{1i}^{\sigma}\}, g_{26}(\boldsymbol{t}_i) = 0, g_{27}(\boldsymbol{t}_i) = 0.
\end{aligned}
$$

(d)

$$
\begin{aligned}
\prod_{i\in B_3}\left(-\frac{\partial S(\boldsymbol{t}_i)}{\partial \boldsymbol{t}_{3i}}\right) = \Bigg[&\prod_{i=1}^n [g_{31}(\boldsymbol{t}_i)]^{r_{1i}}\prod_{i=1}^n [g_{32}(\boldsymbol{t}_i)]^{r_{2i}}\prod_{i=1}^n [g_{33}(\boldsymbol{t}_i)]^{r_{3i}}\prod_{i=1}^n [g_{34}(\boldsymbol{t}_i)]^{r_{4i}}\\
&\times\prod_{i=1}^n [g_{35}(\boldsymbol{t}_i)]^{r_{5i}}\prod_{i=1}^n [g_{36}(\boldsymbol{t}_i)]^{r_{6i}}\prod_{i=1}^n [g_{37}(\boldsymbol{t}_i)]^{r_{7i}}\Bigg]^{(1-\delta_{1i})(1-\delta_{2i})\delta_{3i}},
\end{aligned}
$$

where

$$
\begin{aligned}
g_{31}(\boldsymbol{t}_i) &= \lambda_{3}\sigma t_{3i}^{\sigma-1}\exp\{-\lambda_{14}t_{1i}^{\sigma} - \lambda_2 t_{2i}^{\sigma} - \lambda_3 t_{3i}^{\sigma}\},\\
g_{32}(\boldsymbol{t}_i) &= \lambda_{3}\sigma t_{3i}^{\sigma-1}\exp\{-\lambda_{1}t_{1i}^{\sigma} - \lambda_{24} t_{2i}^{\sigma} - \lambda_3 t_{3i}^{\sigma}\},\\
g_{33}(\boldsymbol{t}_i) &= \lambda_{34}\sigma t_{3i}^{\sigma-1}\exp\{-\lambda_{1}t_{1i}^{\sigma} - \lambda_{2} t_{2i}^{\sigma} - \lambda_{34} t_{3i}^{\sigma}\},\\
g_{34}(\boldsymbol{t}_i) &= 0, g_{35}(\boldsymbol{t}_i) = 0, g_{37}(\boldsymbol{t}_i) = 0, g_{36}(\boldsymbol{t}_i) = \lambda_3\sigma t_{3i}^{\sigma-1}\exp\{-\lambda_3 t_{3i}^{\sigma} - (\lambda-\lambda_3) t_{i}^{\sigma}\},
\end{aligned}
$$

(e)

$$\prod_{i\in B_5}\left(\frac{\partial^2 S(\boldsymbol{t}_i)}{\partial \boldsymbol{t}_{1i}\partial \boldsymbol{t}_{2i}}\right)=\left[\prod_{i=1}^{n}[g_{41}(\boldsymbol{t}_i)]^{r_{1i}}\prod_{i=1}^{n}[g_{42}(\boldsymbol{t}_i)]^{r_{2i}}\prod_{i=1}^{n}[g_{43}(\boldsymbol{t}_i)]^{r_{3i}}\prod_{i=1}^{n}[g_{44}(\boldsymbol{t}_i)]^{r_{4i}}\right.$$
$$\left.\times\prod_{i=1}^{n}[g_{45}(\boldsymbol{t}_i)]^{r_{5i}}\prod_{i=1}^{n}[g_{46}(\boldsymbol{t}_i)]^{r_{6i}}\prod_{i=1}^{n}[g_{47}(\boldsymbol{t}_i)]^{r_{7i}}\right]^{\delta_{1i}\delta_{2i}(1-\delta_{3i})},$$

where

$$\begin{aligned}
g_{41}(\boldsymbol{t}_i)&=\lambda_{14}\lambda_2\sigma^2(t_{1i}t_{2i})^{\sigma-1}\exp\{-\lambda_{14}t_{1i}^{\sigma}-\lambda_2t_{2i}^{\sigma}-\lambda_3t_{3i}^{\sigma}\},\\
g_{42}(\boldsymbol{t}_i)&=\lambda_1\lambda_{24}\sigma^2(t_{1i}t_{2i})^{\sigma-1}\exp\{-\lambda_1t_{1i}^{\sigma}-\lambda_{24}t_{2i}^{\sigma}-\lambda_3t_{3i}^{\sigma}\},\\
g_{43}(\boldsymbol{t}_i)&=\lambda_1\lambda_2\sigma^2(t_{1i}t_{2i})^{\sigma-1}\exp\{-\lambda_1t_{1i}^{\sigma}-\lambda_2t_{2i}^{\sigma}-\lambda_{34}t_{3i}^{\sigma}\},\\
g_{44}(\boldsymbol{t}_i)&=\lambda_1\sigma^2(t_{1i}t_{2i})^{\sigma-1}(\lambda-\lambda_1)\exp\{-\lambda_1t_{1i}^{\sigma}-(\lambda-\lambda_1)t_{2i}^{\sigma}\},\\
g_{45}(\boldsymbol{t}_i)&=(\lambda-\lambda_2)\lambda_2\sigma^2(t_{1i}t_{2i})^{\sigma-1}\exp\{-\lambda_2t_{2i}^{\sigma}-(\lambda-\lambda_2)t_{1i}^{\sigma}\},g_{46}(\boldsymbol{t}_i)=0,g_{47}(\boldsymbol{t}_i)=0.
\end{aligned}$$

(f)

$$\prod_{i\in B_6}\left(\frac{\partial^2 S(\boldsymbol{t}_i)}{\partial \boldsymbol{t}_{1i}\partial \boldsymbol{t}_{3i}}\right)=\left[\prod_{i=1}^{n}[g_{51}(\boldsymbol{t}_i)]^{r_{1i}}\prod_{i=1}^{n}[g_{52}(\boldsymbol{t}_i)]^{r_{2i}}\prod_{i=1}^{n}[g_{53}(\boldsymbol{t}_i)]^{r_{3i}}\prod_{i=1}^{n}[g_{54}(\boldsymbol{t}_i)]^{r_{4i}}\right.$$
$$\left.\times\prod_{i=1}^{n}[g_{55}(\boldsymbol{t}_i)]^{r_{5i}}\prod_{i=1}^{n}[g_{56}(\boldsymbol{t}_i)]^{r_{6i}}\prod_{i=1}^{n}[g_{57}(\boldsymbol{t}_i)]^{r_{7i}}\right]^{\delta_{1i}(1-\delta_{2i})\delta_{3i}},$$

where

$$\begin{aligned}
g_{51}(\boldsymbol{t}_i)&=\lambda_{14}\lambda_3\sigma^2(t_{1i}t_{3i})^{\sigma-1}\exp\{-\lambda_{14}t_{1i}^{\sigma}-\lambda_2t_{2i}^{\sigma}-\lambda_3t_{3i}^{\sigma}\},\\
g_{52}(\boldsymbol{t}_i)&=\lambda_1\lambda_3\sigma^2(t_{1i}t_{3i})^{\sigma-1}\exp\{-\lambda_1t_{1i}^{\sigma}-\lambda_{24}t_{2i}^{\sigma}-\lambda_3t_{3i}^{\sigma}\},\\
g_{53}(\boldsymbol{t}_i)&=\lambda_1\lambda_{34}\sigma^2(t_{1i}t_{3i})^{\sigma-1}\exp\{-\lambda_1t_{1i}^{\sigma}-\lambda_2t_{2i}^{\sigma}-\lambda_{34}t_{3i}^{\sigma}\},\\
g_{54}(\boldsymbol{t}_i)&=,g_{55}(\boldsymbol{t}_i)=0,g_{57}(\boldsymbol{t}_i)=0,\\
g_{56}(\boldsymbol{t}_i)&=(\lambda-\lambda_3)\lambda_3\sigma^2(t_{1i}t_{3i})^{\sigma-1}\exp\{-\lambda_3t_{3i}^{\sigma}-(\lambda-\lambda_2)t_{1i}^{\sigma}\}.
\end{aligned}$$

(g)

$$\prod_{i\in B_7}\left(\frac{\partial^2 S(\boldsymbol{t}_i)}{\partial \boldsymbol{t}_{2i}\partial \boldsymbol{t}_{3i}}\right)=\left[\prod_{i=1}^{n}[g_{61}(\boldsymbol{t}_i)]^{r_{1i}}\prod_{i=1}^{n}[g_{62}(\boldsymbol{t}_i)]^{r_{2i}}\prod_{i=1}^{n}[g_{63}(\boldsymbol{t}_i)]^{r_{3i}}\prod_{i=1}^{n}[g_{64}(\boldsymbol{t}_i)]^{r_{4i}}\right.$$
$$\left.\times\prod_{i=1}^{n}[g_{65}(\boldsymbol{t}_i)]^{r_{5i}}\prod_{i=1}^{n}[g_{66}(\boldsymbol{t}_i)]^{r_{6i}}\prod_{i=1}^{n}[g_{67}(\boldsymbol{t}_i)]^{r_{7i}}\right]^{(1-\delta_{1i})\delta_{2i}\delta_{3i}},$$

where

$$
\begin{aligned}
g_{61}(\boldsymbol{t}_i) &= \lambda_2\lambda_3\sigma^2(t_{2i}t_{3i})^{\sigma-1}\exp\{-\lambda_{14}t_{1i}^{\sigma} - \lambda_2 t_{2i}^{\sigma} - \lambda_3 t_{3i}^{\sigma}\},\\
g_{62}(\boldsymbol{t}_i) &= \lambda_{24}\lambda_3\sigma^2(t_{2i}t_{3i})^{\sigma-1}\exp\{-\lambda_1 t_{1i}^{\sigma} - \lambda_{24} t_{2i}^{\sigma} - \lambda_3 t_{3i}^{\sigma}\},\\
g_{63}(\boldsymbol{t}_i) &= \lambda_2\lambda_{34}\sigma^2(t_{2i}t_{3i})^{\sigma-1}\exp\{-\lambda_1 t_{1i}^{\sigma} - \lambda_2 t_{2i}^{\sigma} - \lambda_{34} t_{3i}^{\sigma}\},\\
g_{64}(\boldsymbol{t}_i) &= 0, g_{65}(\boldsymbol{t}_i) = 0, g_{66}(\boldsymbol{t}_i) = 0, g_{67}(\boldsymbol{t}_i) = 0.
\end{aligned}
$$

(h)

$$
\begin{aligned}
\prod_{i\in B_8} S(\boldsymbol{t}_i) = \Bigg[&\prod_{i=1}^{n}[S_1(\boldsymbol{t}_i)]^{r_{1i}}\prod_{i=1}^{n}[S_2(\boldsymbol{t}_i)]^{r_{2i}}\prod_{i=1}^{n}[S_3(\boldsymbol{t}_i)]^{r_{3i}}\prod_{i=1}^{n}[S_4(\boldsymbol{t}_i)]^{r_{4i}}\\
&\times\prod_{i=1}^{n}[S_5(\boldsymbol{t}_i)]^{r_{5i}}\prod_{i=1}^{n}[S_6(\boldsymbol{t}_i)]^{r_{6i}}\prod_{i=1}^{n}[S_7(\boldsymbol{t}_i)]^{r_{7i}}\Bigg]^{(1-\delta_{1i})(1-\delta_{2i})(1-\delta_{3i})},
\end{aligned}
$$

where $S_1(\boldsymbol{t}_i)$, $S_2(\boldsymbol{t}_i)$, $S_3(\boldsymbol{t}_i)$, $S_4(\boldsymbol{t}_i)$, $S_5(\boldsymbol{t}_i)$, $S_6(\boldsymbol{t}_i)$ and $S_7(\boldsymbol{t}_i)$ are defined by Equation (4).

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