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Nonparametric Statistics
Research Paper

Nonparametric estimation of the relative error in functional regression and censored data

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Abstract

In this paper, the almost complete consistency and the asymptotic normality of the estimator of the regression operator in the case of a censored response given a functional explanatory variable are investigated under some mild conditions. The latter is constructed from the minimization of the mean squared relative error. The novelty of this work compared to the works found in the literature is that the response variable is censored. A simulation study is carried out to compare the finite sample performance based on mean square error between the classical regression and the relative error regression. Moreover, a real data study is used to illustrate our methodology.

Keywords: Censoring · Functional data analysis · Nonparametric statistics · Relative error regression.

Mathematics Subject Classification: Primary 62G05, 62G20 · Secondary 62F12.

1. Introduction

Functional data analysis is a section of statistics that studies the observation of infinite dimension. More precisely, the observations that are not real or vector variables but random curves. This kind of data appears in many practical situations, and it has been the subject of many works. The first authors who discussed this type of data are Ramsay and Silverman (2005) for the parametric models and monograph of Ferraty and Vieu (2006) for the nonparametric estimation. Recently, many topics concerning the analysis of functional data have been developed and the most recent advances in this field have been collected in the book of Ould-Said et al. (2015). The particularity of the nonparametric estimation consists in estimating an infinite number of parameters whose function is unknown, elements of a certain functional class, such as the density function or the regression function. The latter is one of many methods to predict the link between the response variable Y and the explanatory variable X, assuming the existence of a function $r(X)$ which expresses the relationship between these two variables. The literature concerning this field is widely developed. We refer to Ferraty and Vieu (2004) for more details, where is established the strong consistency of the regression function when the response is scalar given a functional

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explanatory variable. Usually, to estimate the nonparametric regression model, the authors used the least squares error as a criterion for constructing the predictors (see some details in Louzada et al. (2018)). This method is very sensitive to outliers, and therefore, the presence of large outliers can lead to inappropriate results. For this, the authors developed methods that study robustness of the nonparametric functional regression; see also Attouch et al. (2009) and Gheriballah et al. (2013).

The relative squared error criterion is more convenient as a measure of performance than the previous criterion, since the notion of relative regression is more recent than the others, although the results are still limited. Jones et al. (2008) studied the asymptotic properties of a consistent estimator of this model by using the kernel method. We refer to Mechab and Laksaci (2016) for recent advances, who studied nonparametric relative regression for associated variables. In a functional framework, the paper of Demongeot et al. (2016) brought an extra to the research by studying the almost complete convergence and asymptotic normality of the proposed estimator.

In this paper, we investigate the asymptotic properties of the relative error regression by the kernel method and under censoring data. The literature of this kind of incomplete functional data is quite restricted. We refer to Kohler et al. (2002) and Horrigue and Ould-Said (2011, 2014) for the nonparametric regression quantile estimation under random censorship. Other works have been conducted on this subject for functional data case. We cite for example the work of Khardani et al. (2010). Moreover, our framework was considered by Altendji et al. (2018) for the estimation of the functional relative error regression under random left truncation, where they established the almost complete convergence with rates, as well as the asymptotic normality of the kernel estimator of the functional relative error regression for truncated data. In a more general field, we can see, for example, Hsing and Eubank (2015) and Aneiros et al. (2017). In the present work, we investigate the almost complete convergence and asymptotic normality of our proposed estimator in case of censored functional data.

The organization of this paper is as follows. In Section 2, we construct an estimator of the relative error regression for a censored response. The necessary conditions and main results are presented in Section 3. In Section 4, a numerical study and a real example show the performances of the proposed methodology for finite samples. Also, we establish a confidence interval as an application for the asymptotic normality result. In Section 5, we provide some concluding remarks. The proofs of our results are given in the appendix.

2. Description of the Model and Estimator

2.1 Estimator of the the relative error regression

Let $Z_i = (X_i, Y_i)_{i=1,\dots,n}$ be a $\mathcal{F} \times \mathrm{R}$ valued measurable strictly stationary process. A common nonparametric modeling of the link between the response variables Y and the explanatory variable X is to suppose that

$$Y = m(X) + \varepsilon, \tag{1}$$

where ε is a random error variable and m is a regression operator usually estimated by minimizing the expected squared loss function given by

$$\mathrm{E}[(Y - m(X))^2 | X].$$

In some situations, this loss function which is considered as a measure of prediction, may not be suitable. Among these situations, the presence of outliers can lead to inappropriate results since all variables have an equal weight. For this, we overcame this limitation by proposing to estimate the function m with respect to the minimization of the mean squared relative error defined as

$$\mathrm{E}\left[\left(\frac{Y-m(X)}{Y}\right)^2 \Big| X\right], \quad Y>0. \tag{2}$$

Obviously, this loss function is a more meaningful measure of prediction performance in the presence of outliers since the range of predicted values is large. Furthermore, the solution of (2) can be expressed by the ratio of first two conditional inverse moments of Y given X. The best predictor of Y given X (as studied in Park and Stefanski (1998)) is given by

$$r(x) = \frac{\mathrm{E}[Y^{-1}|X=x]}{\mathrm{E}[Y^{-2}|X=x]}.$$

We estimate the regression operator r under our relative loss as

$$\widetilde{r}(x) = \frac{\sum_{i=1}^n Y_i^{-1} K(h^{-1}d(x-X_i))}{\sum_{i=1}^n Y_i^{-2} K(h^{-1}d(x-X_i))}, \tag{3}$$

where K is a kernel and $h = h_n$ is a sequence of positive real numbers.

2.2 Estimator of the relative error regression under a random censorship

Let $(X_i, Y_i)_{i=1,\dots,n}$ be a $\mathcal{F} \times \mathrm{R}$ valued measurable strictly stationary process, where $\mathcal{F}$ is a semi-metric abstract space, denote by d, a semi-metric associated with the space $\mathcal{F}$. We observe the lifetimes Y_n as a sequence of independent and identically distributed random variable (with common unknown absolutely continuous distribution function F with density f).

In censoring case, due to possible withdrawals of items from the study, we observe the censored lifetimes C instead observing the lifetimes Y. Supposing that (C_i) is a sequence of independent and identically distributed censoring random variable (r.v.) with common unknown continuous distribution function G. We remark the pairs (T_i, δ_i) where

$$T_i = Y_i \wedge C_i, \quad \delta_i = \mathrm{I}_{\{Y_i \leq C_i\}}, \quad 1 \leq i \leq n,$$

where I_A denotes the indicator of no censoring.

We consider a pseudo estimator of the regression operator r under the censorship and the relative loss given by

$$\widetilde{r}(x) = \frac{\sum_{i=1}^n \delta_i \bar{G}^{-1}(T_i) T_i^{-1} K(h^{-1}d(x-X_i))}{\sum_{i=1}^n \delta_i \bar{G}^{-1}(T_i) T_i^{-2} K(h^{-1}d(x-X_i))} = \frac{\widetilde{g}_1(x)}{\widetilde{g}_2(x)} \tag{4}$$

where $\bar{G}(u) = 1 - G(u)$ and for $l = 1, 2$,

$$\widetilde{g}_l(x) = \frac{1}{n\mathrm{E}(K_1(x))} \sum_{i=1}^n \delta_i \bar{G}^{-1}(T_i) T_i^{-l} K_i(x),$$

where $K_i(x) = K(h^{-1}d(x-X_i))$. Since G is unknown in practice, one can estimate it using the Kaplan and Meier (1958) estimator defined as

$$\bar{G}_n(t) = \begin{cases} \prod_{i=1}^{n} \left(1 - \frac{1-\delta_{(i)}}{n-i+1}\right)^{\mathrm{I}_{\{T_{(i)} \le t\}}} & , \text{if } t < T_{(n)}, \\ 0, & \text{otherwise;} \end{cases}$$

where $T_{(1)} < \cdots < T_{(n)}$ are the order statistics of $(T_i)_{1\le i\le n}$ and $\delta_{(i)}$ is concomitant with $T_{(i)}$. Thus, an estimator of r is given by

$$\widehat{r}_n(x) = \frac{\sum_{i=1}^{n} \delta_i \bar{G}_n^{-1}(T_i) T_i^{-1} K(h^{-1} d(x - X_i))}{\sum_{i=1}^{n} \delta_i \bar{G}_n^{-1}(T_i) T_i^{-2} K(h^{-1} d(x - X_i))} = \frac{\widehat{g}_{1,n}(x)}{\widehat{g}_{2,n}(x)}, \tag{5}$$

where

$$\widehat{g}_{l,n}(x) = \frac{1}{n\mathrm{E}(K_1(x))} \sum_{i=1}^{n} \delta_i \bar{G}_n^{-1}(T_i) T_i^{-l} K_i(x), \quad l = 1, 2.$$

Let $\tau_F = \sup\{y, \bar{F}(y) > 0\}$ and $\tau_G = \sup\{y, \bar{G}(y) > 0\}$ be a upper endpoints of $\bar{F}$ and $\bar{G}$, respectively. We assume that $\tau_F < \infty$, $\bar{G}(\tau_F) > 0$, which implies that $\tau_F \le \tau_G$) and that $(C_n)_{n\ge1}$ and $(X_n, Y_n)_{n\ge1}$ are independent.

3. Assumptions and Main Results

3.1 Consistency: almost complete convergence

We fixe a point x in $\mathcal{F}$ and N_x denotes a fixed neighborhood of this point. We will denote by C and $C^{'}$ some strictly positive constants, $g_l(x) = \mathrm{E}[Y^{-l}|X = x]$ for $l = 1, 2$ and we have $B(x, h) = \{x^{'} \in \mathcal{F} | d(x^{'}, x) < h\}$ a ball of center x and a radius h. In what follows, we will need the following assumptions:

(H1) For all $h > 0$, $\mathrm{P}(X \in B(x, h)) =: \phi_x(h) > 0$ and $\lim_{h\to0} \phi_x(h) = 0$.

(H2) For all $(x_1, x_2) \in N_x^2$ and $l = 1, 2$, we have

$$|g_l(x_1) - g_l(x_2)| \le C d^{k_l}(x_1, x_2) \quad \text{for} \quad k_l > 0.$$

(H3) The kernel K is a measurable function that is supported by (0, 1) and satisfies:

$$0 < C \le K \le C^{'} < \infty.$$

(H4) The bandwidth satisfies:

$$\lim_{n\to+\infty} h = 0 \;\; \text{and} \;\; \lim_{n\to+\infty} \frac{\log(n)}{n\phi_x(h)} = 0.$$

(H5) The inverse moments of the response variable verify:

$$\mathrm{E}[Y^{-m}|X = x] < C < \infty, \quad \forall m \ge 2.$$

Remark 1 The hypothesis (H1) defines the concentration properties of the probability measures of the explanatory variable X, which is provided by means of a function ϕ_x. This property allows to propose an alternative to the curse of dimensionality problem. (H2) is a regularity condition to facilitate the calculation of the bias part of our estimator. (H3)-(H5) are technical assumptions to ensure the convergence of our results.

THEOREM 3.1 Assume that conditions (H1)-(H5) hold true, we get

$$|\widehat{r}_n(x) - r(x)| = O(h^{k_1}) + O(h^{k_2}) + O\left(\sqrt{\frac{\log(n)}{n\phi_x(h)}}\right). \tag{6}$$

LEMMA 3.2 Under assumptions (H1)-(H4), we obtain, for $l = 1, 2$,

$$|\mathrm{E}\left[\widetilde{g}_l(x)\right] - g_l(x)| = O(h^{k_l}). \tag{7}$$

LEMMA 3.3 Under conditions (H1) and (H3)-(H5), we have, for $l = 1, 2$,

$$|\widetilde{g}_l(x) - \mathrm{E}\left[\widetilde{g}_l(x)\right]| = O\left(\sqrt{\frac{\log(n)}{n\phi_x(h)}}\right). \tag{8}$$

LEMMA 3.4 Assume hypotheses (H1)-(H5) hold, we have, for $l = 1, 2$,

$$|\widehat{g}_{l,n}(x) - \widetilde{g}_l(x)| = O_{a.s}\left(\sqrt{\frac{\log(\log(n))}{n}}\right). \tag{9}$$

COROLLARY 3.5 Under assumptions of Theorem 3.1,we get

$$|\widehat{g}_{2,n}(x)| \xrightarrow[n\to\infty]{} g_2(x).$$

3.2 Asymptotic normality

Here, we establish the asymptotic normality of the estimator $\widehat{r}_n(x)$. To do that, we consider the following assumptions:

(C1) The hypothesis (H1) holds and there exists a function χ_x such that, for all $s \in [0, 1]$, we have $\phi_x(sr)/\phi_x(r) = \chi_x(s) + o(1)$ and $\int_0^1 (K^j)'(s)\chi_x(s)\mathrm{d}s < \infty$, for $j \geq 1$.
(C2) The functions $\Psi_l(u) = \mathrm{E}[g_l(X) - g_l(x)|d(x, X) = u]$ are derivable at 0, for $l = 1, 2$.
(C3) The hypothesis (H3) holds and the kernel K is a differentiable function on $]0, 1[$ and its first derivative function K' satisfies that $C < K' < C'$.
(C4) The small ball probability satisfies:

$$n\phi_x(h) \to \infty.$$

(C5) The inverse moments $g_m(u) = \mathrm{E}[|\bar{G}^{-1}(Y)Y^{-m}||X = u]$ of the censored response variable are continuous in a neighborhood of x, for $m = 1, 2, 3, 4$.

Remark 2 The condition $(C1)$ is realized by several small ball probability functions, there exist many examples, we quote the following (which can be found in Ferraty et al. (2007)):

(i) For some $\gamma > 0$, $\phi_x(h) = C_x h^\gamma$ with $\chi_x(u) = u^\gamma$,

(ii) for some $\gamma > 0$ and $p > 0$, $\phi_x(h) = C_x h^\gamma \exp(-C/h^p)$, with $\chi_x(u) = \delta_1(u)$, where δ_1 is the Dirac function,

(iii) $\phi_x(h) = C_x/|\log(h)|$, with $\chi_x(u) = \mathrm{I}_{[0,1]}(u)$, where I_A is an indicator function of a set A.

THEOREM 3.6 Suppose that conditions (C1)-(C5) hold true, for all $x \in \mathcal{F}$, we have, as $n \to \infty$,

$$\left(\frac{n\phi_x(h)}{\sigma^2(x)}\right)^{\frac{1}{2}} (\widehat{r}_n(x) - r(x)) \xrightarrow{\mathcal{D}} \mathrm{N}(0,1),$$

where $\xrightarrow{\mathcal{D}}$ means the convergence in distribution and

$$\sigma^2(x) = \frac{M_2}{M_1^2}\left(g_2(x) + r^2(x)g_4(x) - 2r(x)g_3(x)\right),$$

with $M_0 = K(1) - \int_0^1 (sK(s))'\chi_x(s)\mathrm{d}s$ and $M_j = K^j(1) - \int_0^1 (K^j)'(s)\chi_x(s)\mathrm{d}s$, for $j = 1, 2$.

PROOF OF THEOREM 3.6. From the decomposition 10, we get the decomposition

$$\begin{aligned}
\widehat{r}_n(x) - r(x) = \frac{1}{\widehat{g}_{2,n}(x)g_2(x)} & \left[\left(\widetilde{g}_1(x) - \mathrm{E}\left[\widetilde{g}_1(x)\right]\right) g_2(x) + \left(\mathrm{E}\left[\widetilde{g}_2(x)\right] - \widetilde{g}_2(x)\right) g_1(x)\right. \\
& + \left(\widehat{g}_{1,n}(x) - \widetilde{g}_1(x)\right) g_2(x) + \left(\widetilde{g}_2(x) - \widehat{g}_{2,n}(x)\right) g_1(x) \\
& \left. + \left(\mathrm{E}\left[\widetilde{g}_1(x)\right] - g_1(x)\right) g_2(x) + \left(g_2(x) - \mathrm{E}\left[\widetilde{g}_2(x)\right]\right) g_1(x)\right].
\end{aligned}$$

Then, Theorem 3.6 is a consequence of the following lemmas.

LEMMA 3.7 Under the same conditions of Theorem 3.6, we have

$$\left(\frac{n\phi_x(h)}{g_2^2(x)\sigma^2(x)}\right)^{\frac{1}{2}} \left(\left[\widetilde{g}_1(x) - \mathrm{E}\left[\widetilde{g}_1(x)\right]\right] g_2(x) + \left[\mathrm{E}\left[\widetilde{g}_2(x)\right] - \widetilde{g}_2(x)\right] g_1(x)\right) \xrightarrow{\mathcal{D}} \mathrm{N}(0,1).$$

LEMMA 3.8 Under hypotheses of Theorem 3.6, we get $\widehat{g}_{2,n}(x) \to g_2(x)$, in probability, and

$$\left(\frac{n\phi_x(h)}{g_2^2(x)\sigma^2(x)}\right)^{\frac{1}{2}} \left[\left(\widehat{g}_{1,n}(x) - \widetilde{g}_1(x)\right) g_2(x) + \left(\widetilde{g}_2(x) - \widehat{g}_{2,n}(x)\right) g_1(x)\right] \to 0,$$

in probability.

LEMMA 3.9 Under hypotheses of Theorem 3.6, we obtain

$$\left(\frac{n\phi_x(h)}{g_2^2(x)\sigma^2(x)}\right)^{\frac{1}{2}} \left[\frac{1}{g_2(x)}\left(\mathrm{E}\left[\widetilde{g}_1(x)\right] - g_1(x)\right) g_2(x) + \left(g_2(x) - \mathrm{E}\left[\widetilde{g}_2(x)\right]\right) g_1(x)\right] \to 0,$$

in probability.

4. Numerical Studies

4.1 Simulation study on the finite samples

To compare the finite-sample performance of the proposed estimator of $r(x) = \mathrm{E}[Y|X = x]$ to the classical regression, we conducted a small simulation study. We consider a functional regression model defined as

$$Y_i = m(X_i) + \varepsilon,$$

where the random variable ε is normally distributed as $\mathrm{N}(0, 1)$ and

$$m(x) = 4 \exp\left(\frac{1}{1 + \int_0^{\pi} |x(t)|^2 \mathrm{d}t}\right).$$

The functional variable X is chosen as a real-valued function with support $[0, \pi]$, we generate $n = 100$ functional data (see Figure 1) by $X_i(t) = \sin(W_i(t))$, for all $t \in [0, \pi]$ and $i = 1, \ldots, n$, where the random variables W_i are independent and identically distributed and follow the normal distribution $\mathrm{N}(0, 1)$. The curves are discretized on the same grid which is composed of 100 equidistant values in $[0, \pi]$.

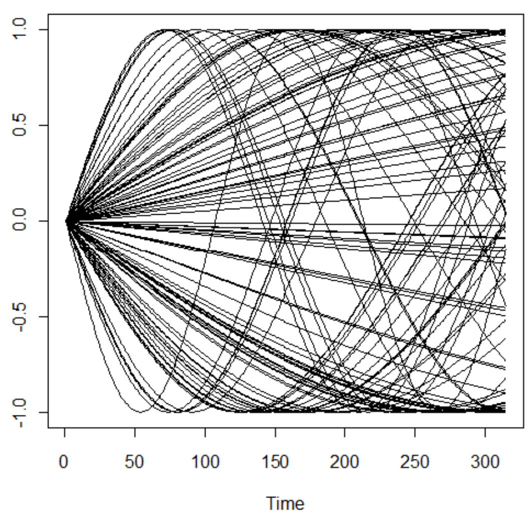

Figure 1. Curves X_i

Our purpose is to compare the mean square error (MSE) of the estimator of relative error regression (RER) with the censored data set and with the classical regression estimator (CR) respectively which are defined as

$$\widehat{r}_n(x) = \frac{\sum_{i=1}^{n} \delta_i \bar{G}_n^{-1}(T_i) T_i^{-1} K(h^{-1} d(x - X_i))}{\sum_{i=1}^{n} \delta_i \bar{G}_n^{-1}(T_i) T_i^{-2} K(h^{-1} d(x - X_i))}$$

and

$$\widehat{r}(x) = \frac{\sum_{i=1}^{n} \delta_i \bar{G}_n^{-1}(T_i) T_i K(h^{-1} d(x - X_i))}{\sum_{i=1}^{n} \delta_i \bar{G}_n^{-1}(T_i) K(h^{-1} d(x - X_i))}.$$

We choose the quadratic kernel given by

$$K(u) = \frac{3}{4}(1 - u^2)\mathrm{I}_{[-1,1]}(u)$$

and the bandwidth h is automatically selected by the procedure of the cross validation.

We give the formula of the MSEs of the both estimators as

$$\mathrm{MSE(RER)} = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \widehat{r}_{n,i}(X_i))^2$$

and

$$\mathrm{MSE(CR)} = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \widehat{r}_i(X_i))^2,$$

where $\widehat{r}_{n,i}$ $(\widehat{r}_i)$ is the leave-one-out version of $\widehat{r}_n$ $(\widehat{r})$ computed by removing the ith data from the initial sample.

Table 1. Values of the MSE according to the number of introduced artificial outliers (first line).

Outliers	5	10	20	30	40	50
CR	0.5254138	70.67035	658.129	3702.399	5923.839	14809.60
RER	0.1219565	0.1256098	0.1261814	0.1261834	0.1261834	0.1261834

Note from Table 1 that the MSE values for both kernel methods increase considerably relative to the presence of the outliers, while these errors remain very small in the case of the relative error estimator. In conclusion, the relative error regression performs better than the classical regression, that is, the classical regression is more sensitive to the presence of outliers than the relative error regression.

4.2 Real data application

We apply the theoretical results obtained in the previous section to real data. More specifically, we examine the performance of the relative regression estimator in the presence of outliers than the classical kernel method. For this purpose application, we consider the spectroscopic dataset, are available from `http://www.models.kvl.dk/NIRsoil`. The data concern spectra of 108 soil samples measured by near infrared reflectance (NIR), in the range 400–2500 nanometre (nm) with a 2 nm resolution (Rinnan and Rinnan, 2007). Thus, the soil samples are obtained during a long-term climate change manipulation experiment at a subarctic fell heath in Abisko, northern Sweden. Moreover, to determine the chemical and microbiological properties of soil, soil organic matter (SOM) was measured as loss on ignition at 550°C and ergosterol concentration was determined through High-Performance Liquid Chromatography (HPLC), which are taken in the following as two response variables. The aim is to analyse relationships between the NIR data (X-variables), and the chemical and microbiological data (Y-variables). For each sample soil, one observes a spectroscopic curve which corresponds to the reflectance at 1050 wavelengths, and its soil organic matter and ergosterol content. Hence, $X_i(t)$ is the reflectance of the i^{th} sample of soil at wavelength t, where $t \in \{400, \ldots, 2500\}$. Let Y_1 and Y_2 be two response variables which correspond to soil organic matter and ergosterol concentration, respectively (see Figures 3 and 4). The functional covariates in Figure 2 shows the 108 NIR reflectance spectra.

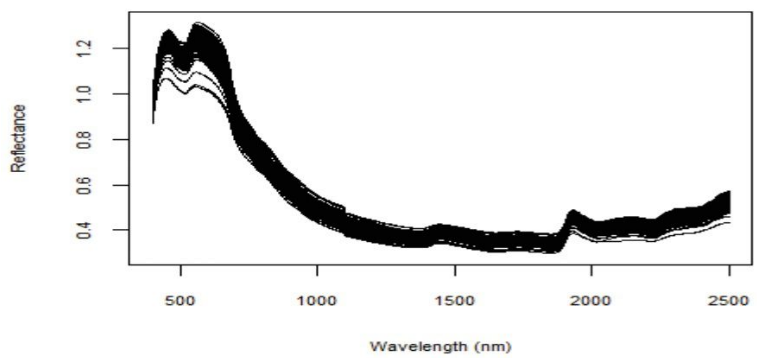

Figure 2. Curves of 108 NIR spectra

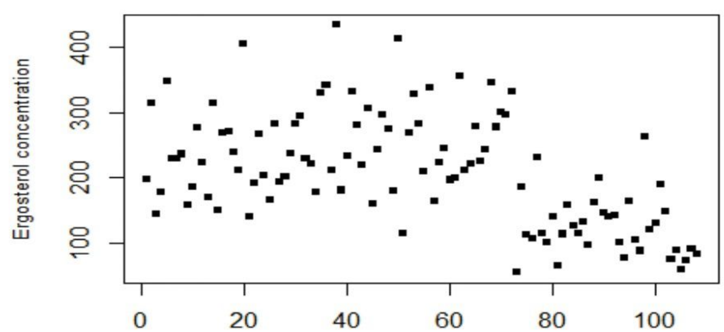

Figure 3. The distribution of 108 values of Y_1 (SOM)

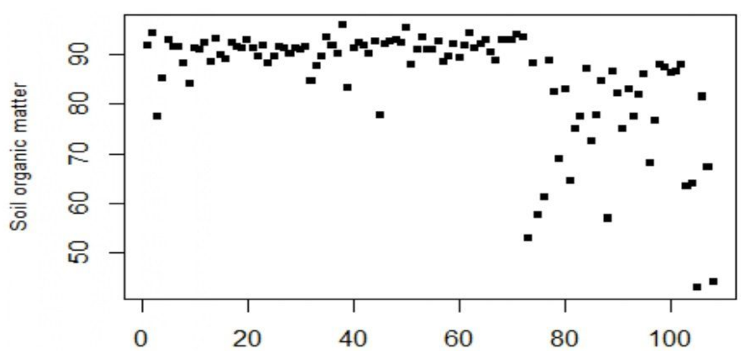

Figure 4. The distribution of 108 values of Y_2 (ergosterol concentration)

Applied to NIR data the MAD-Median method identifies 21 outliers for Y_1 and 1 outlier for Y_2. Recall that we are interested to build two models: $Y_1 = r_1(X) + \varepsilon_1$ and $Y_2 = r_2(X) + \varepsilon_2$, where $r_1(x) = \mathrm{E}(Y_1|X = x)$ and $r_2(x) = \mathrm{E}(Y_2|X = x)$. Furthermore, the dataset was randomly split into a learning sample (72 curves) used to build the estimators, and a testing sample (36 curves) which allows computing the MSE. We note that the result of our simulation study is evaluated over 100 independent replications and its sensitivity to grid sizes or to size of test sample and training sample is not very substantial. Because of the smoothness of the NIR curves, we use the semi-metric based on the second order derivatives, where the curves are replaced by their B-spline expansion. Here, the best results in terms of prediction are obtained for a number of interior knots needed for defining the B-spline basis, equal to 40. Therefore, we chosen the smoothing parameter h via a local cross-validation method on the number of nearest-neighbors. It can be seen that, in the presence of outliers, the relative regression estimator performs better than the classical kernel method. This is confirmed by the MSE obtained respectively in the two cases of study.

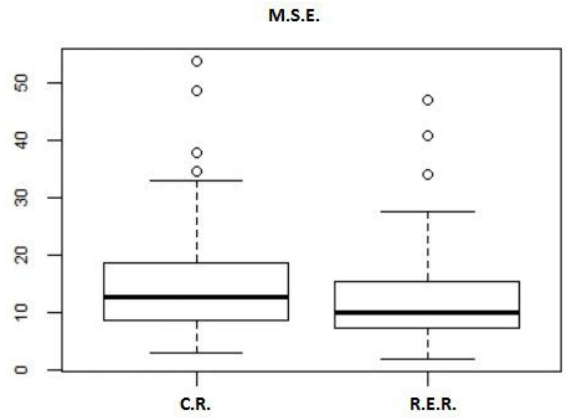

Figure 5. Box plots of the MSE for Y_1

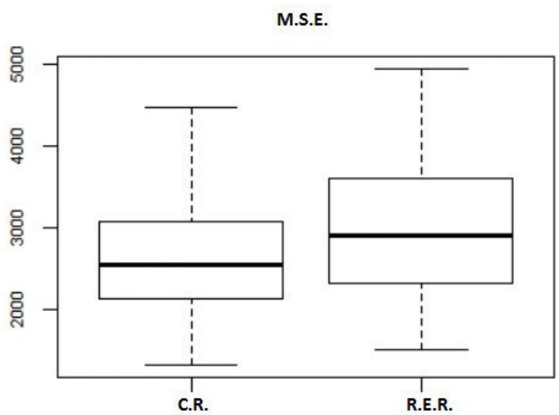

Figure 6. Box plots of the MSE for Y_2

4.3 Confidence bands

A usual application of asymptotic normality is to establish confidence intervals for the true value of the proposed estimator. To determine this band, we need the estimation of the unknown quantity of the asymptotic variance. In our case, we have

$$\sigma^2(x) = \frac{M_2}{M_1^2}\left(g_2(x) + r^2(x)g_4(x) - 2r(x)g_3(x)\right),$$

where M_1, M_2, r and g_l, for $l = 1, 2, 3, 4$, are unknown in practice and have to be estimated. Now a plug-in estimate for the asymptotic standard deviation $\sigma(x)$ can be easily obtained using the estimators $\widehat{M}_1, \widehat{M}_2, \widehat{r}_n$ and $\widehat{g}_{l,n}$ of M_1, M_2, r and g_l respectively. Precisely, we estimate $g_3(x)$ and $g_4(x)$ in the same way as for $g_1(x)$ and $g_2(x)$.

We estimate empirically the constants M_1 and M_2, as

$$\widehat{M}_1 = \frac{1}{n\phi_x(h)} \sum_{i=1}^{n} \delta_i \bar{G}_n^{-1}(T_i) K_i(x)$$

and

$$\widehat{M}_2 = \frac{1}{n\phi_x(h)} \sum_{i=1}^{n} \delta_i \bar{G}_n^{-1}(T_i) K_i^2(x).$$

Furthermore, we get

$$\widehat{\sigma}(x) = \left(\frac{\widehat{M}_2}{\widehat{M}_1^2} \left(\widehat{g}_{2,n}(x) + \widehat{r}_n^2(x) \widehat{g}_{4,n}(x) - 2\widehat{r}_n(x) \widehat{g}_{3,n}(x) \right) \right)^{\frac{1}{2}}.$$

We have approximate $(1-\zeta)$ confidence bands for $r(x)$ given by

$$\left[\widehat{r}_n(x) - t_{1-\frac{\zeta}{2}} \left(\frac{\widehat{\sigma}^2(x)}{n\phi_x(h)} \right)^{\frac{1}{2}}, \quad \widehat{r}_n(x) + t_{1-\frac{\zeta}{2}} \left(\frac{\widehat{\sigma}^2(x)}{n\phi_x(h)} \right)^{\frac{1}{2}} \right],$$

where $t_{1-\frac{\zeta}{2}}$ denotes the $1 - \frac{\zeta}{2} \times 100$th quantile of the standard normal distribution.

5. Concluding Remarks

This paper illustrated the asymptotic properties of the regression operator estimator based on the minimization of the mean squared relative error under censoring data. The resulting relative error regression showed to be consistent and asymptotically distributed normally under appropriate conditions in case of censored functional data. Our theoretical and practical studies confirmed that the relative error regression is more efficient than the classical regression.

Appendix

Proof of Theorem 3.1. This is based on the following decomposition

$$\begin{aligned} |\widehat{r}_n(x) - r(x)| = {}& \frac{1}{\widehat{g}_{2,n}(x)} \left[|\widehat{g}_{1,n}(x) - \widetilde{g}_1(x)| + |\widetilde{g}_1(x) - \mathrm{E}\left[\widetilde{g}_1(x)\right]| + |\mathrm{E}\left[\widetilde{g}_1(x)\right] - g_1(x)| \right] \\ &+ \frac{r(x)}{\widehat{g}_{2,n}(x)} \left[|\widetilde{g}_2(x) - \widehat{g}_{2,n}(x)| + |\mathrm{E}\left[\widetilde{g}_2(x)\right] - \widetilde{g}_2(x)| + |g_2(x) - \mathrm{E}\left[\widetilde{g}_2(x)\right]| \right]. \quad (10) \end{aligned}$$

Thus, we prove Theorem 3.1 by the following intermediate results

Proof of Lemma 3.2. We have

$$|\mathrm{E}\left[\widetilde{g}_l(x)\right] - g_l(x)| = \left| \frac{1}{n\mathrm{E}[K_1(x)]} \sum_{i=1}^{n} \mathrm{E}[\delta_i \bar{G}^{-1}(T_i) T_i^{-l} K_i(x) - g_l(x)] \right|.$$

By using a double conditioning with respect to Y_i, we get

$$\begin{aligned}
\mathrm{E}\left[\widetilde{g}_l(x)\right] &= \frac{1}{n\mathrm{E}[K_1(x)]}\sum_{i=1}^{n}\mathrm{E}[\mathrm{E}(\delta_i\bar{G}^{-1}(T_i)T_i^{-l}K_i(x)|X_i)] \\
&= \frac{1}{\mathrm{E}[K_1(x)]}\mathrm{E}[K(h^{-1}(x-X_1))\mathrm{E}(\delta_1\bar{G}^{-1}(T_1)T_1^{-l}|X_1)] \\
&= \frac{1}{\mathrm{E}[K_1(x)]}\mathrm{E}[K(h^{-1}(x-X_1))\mathrm{E}(\mathrm{E}[\delta_1\bar{G}^{-1}(T_1)T_1^{-l}|Y_1]|X_1)] \\
&= \frac{1}{\mathrm{E}[K_1(x)]}\mathrm{E}[K(h^{-1}(x-X_1))\mathrm{E}(\bar{G}^{-1}(Y_1)Y_1^{-l}\mathrm{E}[\mathrm{I}_{\{Y_1\leq C_1\}}|Y_1]|X_1)].
\end{aligned}$$

Then,

$$\begin{aligned}
\mathrm{E}\left[\widetilde{g}_l(x)-g_l(x)\right] &= \frac{1}{\mathrm{E}[K_1(x)]}\mathrm{E}\left[K(h^{-1}(x-X_1))\mathrm{I}_{B(x,h)}(X_1)\left|\mathrm{E}(Y_1^{-l}|X_1)-g_l(x)\right|\right] \\
&= \frac{1}{\mathrm{E}[K_1(x)]}\mathrm{E}\left[K(h^{-1}(x-X_1))\mathrm{I}_{B(x,h)}(X_1)|g_l(X_1)-g_l(x)|\right].
\end{aligned}$$

Thus, under conditions (H2), we get

$$\begin{aligned}
\left|\mathrm{E}\left[\widetilde{g}_l(x)-g_l(x)\right]\right| &\leq Ch^{k_l} \\
&= O(h^{k_l}).
\end{aligned}$$

Proof of Lemma 3.3. We have for $l=1,2$

$$\widetilde{g}_l(x)-\mathrm{E}\left[\widetilde{g}_l(x)\right] = \frac{1}{n\mathrm{E}[K_1(x)]}\sum_{i=1}^{n}\left[\delta_i\bar{G}^{-1}(T_i)T_i^{-l}K_i(x)-\mathrm{E}[\delta_i\bar{G}^{-1}(T_i)T_i^{-l}K_i(x)]\right].$$

Now, we consider

$$Z_{i,l} = \frac{1}{\mathrm{E}[K_1(x)]}\left[\delta_i\bar{G}^{-1}(T_i)T_i^{-l}K_i(x)-\mathrm{E}[\delta_i\bar{G}^{-1}(T_i)T_i^{-l}K_i(x)]\right].$$

To prove this lemma, we use the exponential inequality given in the monograph of Ferraty and Vieu (2006) (Corollary A.8i). We calculate the quantity of $\mathrm{E}[|Z_{i,l}^m|]$ similarly as in Lemma 6.3 of Ferraty and Vieu (2006). By the Newton binomial expansion, we get

$$\begin{aligned}
\mathrm{E}[|Z_{i,l}^m|] &\leq C\sum_{j=0}^{m}\frac{1}{(\mathrm{E}[K_1])^j}\mathrm{E}\left[\left|\delta_1\bar{G}^{-j}(T_1)T_1^{-jl}K_1^j(x)\right|\right] \\
&\leq C\max_{j=0,\dots,m}\phi_x^{-j+1}(h) \\
&\leq C\phi_x^{-m+1}(h).
\end{aligned}$$

Then,

$$\mathrm{E}[|Z_{i,l}^m|] = O(\phi_x^{-m+1}(h)).$$

Thus, by applying the mentioned exponential inequality with $a^2 = \phi_x^{-1}(h)$, we have, for all $\varepsilon > 0$,

$$\mathrm{P}\left(\left|\sum_{i=1}^{n} Z_{i,l}\right| > \varepsilon n\right) \leq 2\exp\left(\frac{-\varepsilon^2 n}{2a^2(1+\varepsilon)}\right).$$

We establish

$$\varepsilon = \varepsilon_0\sqrt{\frac{\log(n)}{n\phi_x(h)}}.$$

Hence,

$$\begin{aligned}
\mathrm{P}\left(\left|\sum_{i=1}^{n} Z_{i,l}\right| > \varepsilon n\right) &\leq 2\exp\left(\frac{-\varepsilon_0^2\frac{\log(n)}{n\phi_x(h)}n}{2\frac{1}{\phi_x(h)}(1+\varepsilon_0\sqrt{\frac{\log(n)}{n\phi_x(h)}})}\right)\\
&\leq 2\exp\left(-\frac{\varepsilon_0^2\log(n)}{2(1+\varepsilon_0\sqrt{\frac{\log(n)}{n\phi_x(h)}})}\right)\\
&\leq 2\exp\left(-C\varepsilon_0^2\log(n)\right)\\
&\leq 2n^{-C\varepsilon_0^2}.
\end{aligned}$$

Therefore, an appropriate choice of ε_0 and by Proposition A.4. in Ferraty and Vieu (2006), we deduce that

$$|\widetilde{g}_l(x) - \mathrm{E}[\widetilde{g}_l(x)]| = O\left(\sqrt{\frac{\log(n)}{n\phi_x(h)}}\right) = o(1).$$

Proof of Lemma 3.4. We have

$$\begin{aligned}
|\widehat{g}_{l,n}(x) - \widetilde{g}_l(x)| &= \left|\frac{1}{n\mathrm{E}[K_1(x)]}\sum_{i=1}^{n}\delta_i\bar{G}_n^{-1}(T_i)T_i^{-l}K\left(\frac{x-X_i}{h}\right) - \right.\\
&\qquad\qquad \left.\delta_i\bar{G}^{-1}(T_i)T_i^{-l}K\left(\frac{x-X_i}{h}\right)\right|\\
&= \frac{1}{n\mathrm{E}[K_1(x)]}\sum_{i=1}^{n}\left|\mathrm{I}_{\{Y_i\leq C_i\}}\bar{G}_n^{-1}(Y_i)Y_i^{-l}K\left(\frac{x-X_i}{h}\right) - \right.\\
&\qquad\qquad \left.\mathrm{I}_{\{Y_i\leq C_i\}}\bar{G}^{-1}(Y_i)Y_i^{-l}K\left(\frac{x-X_i}{h}\right)\right|\\
&\leq \frac{1}{n\mathrm{E}[K_1(x)]}\sum_{i=1}^{n}\left|Y_i^{-l}K\left(\frac{x-X_i}{h}\right)\left(\frac{1}{\bar{G}_n(Y_i)} - \frac{1}{\bar{G}(Y_i)}\right)\right|\\
&\leq \frac{\sup_{t\leq t_F}|\bar{G}_n(t) - \bar{G}(t)|}{\bar{G}_n(t_F)\bar{G}(t_F)}\frac{1}{n\mathrm{E}[K_1(x)]}\sum_{i=1}^{n}Y_i^{-l}K\left(\frac{x-X_i}{h}\right).
\end{aligned}$$

By using conditional expectation, we obtain

$$|\widehat{g}_{l,n}(x) - \widetilde{g}_l(x)| \leq \frac{\sup_{t\leq t_F} |\bar{G}_n(t) - \bar{G}(t)|}{\bar{G}_n(t_F)\bar{G}(t_F)} \frac{1}{n\mathrm{E}[K_1(x)]} \sum_{i=1}^{n} \mathrm{E}\left[Y_i^{-l} K\left(\frac{x - X_i}{h}\right) |X_i\right].$$

Under conditions (H3), (H5) and by taking into account formula (4.28) in Deheuvels and Einmahl (2000), we get

$$|\widehat{g}_{l,n}(x) - \widetilde{g}_l(x)| = O\left(\frac{\log(\log(n))}{n}\right).$$

Proof of Corollary 3.5. We have

$$\mathrm{P}\left(\lim_{n\to\infty} \widehat{g}_{2,n}(x) = g_2(x)\right) = 1.$$

By taking into account the results of Lemmas 3.2-3.4, we prove the corollary.

Proof of Lemma 3.7. We use the same arguments as in Lemma 7 of Demongeot et al. (2016) for censored data.

Let

$$\frac{\sqrt{n\phi_x(h)}}{g_2^2(x)\sigma(x)} \left(\left[\widetilde{g}_1(x) - \mathrm{E}\left[\widetilde{g}_1(x)\right]\right] g_2(x) + \left[\mathrm{E}\left[\widetilde{g}_2(x)\right] - \widetilde{g}_2(x)\right] g_1(x)\right) = \frac{S_n}{g_2^2(x)\sigma(x)},$$

with $S_n = \sum_{i=1}^{n} \left(L_i(x) - \mathrm{E}[L_i(x)]\right)$, where

$$L_i(x) = \frac{\sqrt{n\phi_x(h)}}{n\mathrm{E}[K_1]} \delta_i \bar{G}^{-1}(T_i) K_i(x) \left(g_1(x) T_i^{-2} - g_2(x) T_i^{-1}\right).$$

We apply the Lyapunov central limit theorem on $L_i(x)$ for showing the asymptotic normality of S_n. It suffices to show, for some $\delta > 0$, that

$$\frac{\sum_{i=1}^{n} \mathrm{E}\left[|L_i(x) - \mathrm{E}[L_i(x)]|^{2+\delta}\right]}{\left(var\left(\sum_{i=1}^{n} L_i(x)\right)\right)^{\frac{2+\delta}{2}}} \to 0. \tag{11}$$

Clearly,

$$\begin{aligned}
\mathrm{Var}\left(\sum_{i=1}^{n} L_i(x)\right) &= n\phi_x(h) \mathrm{Var}\left[\widetilde{g}_2(x) g_1(x) - \widetilde{g}_1(x) g_2(x)\right] \\
&= n\phi_x(h) \left[\mathrm{Var}\left(\widetilde{g}_2(x)\right) g_1^2(x) + \mathrm{Var}\left(\widetilde{g}_1(x)\right) g_2^2(x) - 2 g_1(x) g_2(x) \mathrm{Cov}(\widetilde{g}_1(x), \widetilde{g}_2(x))\right].
\end{aligned}$$

Thus, for $l = 1, 2$, we obtain

$$\begin{aligned}
\mathrm{Var}\left(\widetilde{g}_l(x)\right) &= \frac{1}{(n\mathrm{E}[K_1])^2} \sum_{i=1}^{n} \mathrm{Var}\left[\delta_i \bar{G}^{-1}(T_i) T_i^{-l} K_i(x)\right] \\
&= \frac{1}{n(\mathrm{E}[K_1])^2} \mathrm{Var}\left[\delta_1 \bar{G}^{-1}(T_1) T_1^{-l} K_1(x)\right].
\end{aligned}$$

By conditioning on the random variable X, using hypotheses $(C1)$ and $(C3)$ and the fact that

$$\mathrm{E}[K_1] = \phi_x(h)\left(K(1) - \int_0^1 K^{'}(s)\chi_x(s)\mathrm{d}s\right) + o(\phi_x(h)),$$

we get

$$\begin{aligned}\mathrm{E}\left[\delta_1 \bar{G}^{-2}(T_1)T_1^{-2l}K_1^2(x)\right] &= \mathrm{E}\left[K_1^2(x)\mathrm{E}\left[\bar{G}^{-1}(Y)Y^{-2l}|X=x\right]\right] \\ &= \mathrm{E}\left[\bar{G}^{-1}(Y)Y^{-2l}|X=x\right] \\ &\quad \times \left(\phi_x(h)\left(K^2(1) - \int_0^1 (K^2)^{'}(s)\chi_x(s)\mathrm{d}s\right) + o(\phi_x(h))\right).\end{aligned}$$

By a double conditioning on the random variable X and under conditions (H3) and (H5), we obtain

$$\begin{aligned}\mathrm{E}\left[\delta_1 \bar{G}^{-1}(T_1)T_1^{-l}K_1(x)\right] &= \mathrm{E}\left[K_1(x)\mathrm{E}\left[\mathrm{Y}_1^{-1}|X=x\right]\right] \\ &\le C\mathrm{E}[K_1] \\ &\le C\phi_x(h).\end{aligned}$$

Therefore,

$$\left(\mathrm{E}\left[\delta_1 \bar{G}^{-1}(T_1)T_1^{-l}K_1(x)\right]\right)^2 = O(\phi_x(h)^2).$$

Then,

$$\begin{aligned}\mathrm{Var}\left[\delta_1 \bar{G}^{-1}(T_1)T_1^{-l}K_1(x)\right] &= \mathrm{E}\left[\bar{G}^{-1}(Y)Y^{-2l}|X=x\right] \\ &\quad \times \left(\phi_x(h)\left(K^2(1) - \int_0^1 (K^2)^{'}(s)\chi_x(s)\mathrm{d}s\right)\right) + O(\phi_x(h)^2).\end{aligned}$$

Thus,

$$\mathrm{Var}\left(\widetilde{g}_l(x)\right) = \frac{\mathrm{E}\left[\bar{G}^{-1}(Y)Y^{-2l}|X=x\right]\left(K^2(1) - \int_0^1 (K^2)^{'}(s)\chi_x(s)\mathrm{d}s\right)}{n\phi_x(h)\left(K(1) - \int_0^1 K^{'}(s)\chi_x(s)\mathrm{d}s\right)^2} \tag{12}$$

$$+ \; o\left(\frac{1}{n\phi_x(h)}\right). \tag{13}$$

Now, we calculate the corresponding covariance as

$$\begin{aligned}\mathrm{Cov}(\widetilde{g}_1(x), \widetilde{g}_2(x)) &= \frac{1}{n(\mathrm{E}[K_1])^2}\mathrm{Cov}\left(\delta_1 \bar{G}^{-1}(T_1)T_1^{-1}K_1(x), \delta_1 \bar{G}^{-1}(T_1)T_1^{-2}K_1(x)\right) \\ &= \frac{1}{n(\mathrm{E}[K_1])^2}\left[\mathrm{E}\left(\delta_1 \bar{G}^{-2}(T_1)T_1^{-3}K_1^2(x)\right)\right. \\ &\quad \left. - \mathrm{E}\left(\delta_1 \bar{G}^{-1}(T_1)T_1^{-1}K_1(x)\right)\mathrm{E}\left(\delta_1 \bar{G}^{-1}(T_1)T_1^{-2}K_1(x)\right)\right]\end{aligned}$$

where

$$\mathrm{E}\left(\delta_1\bar{G}^{-2}(T_1)T_1^{-3}K_1^2(x)\right) = \mathrm{E}\left[K_1^2\mathrm{E}\left[\bar{G}^{-1}Y^{-3}|X=x\right]\right]$$
$$= \mathrm{E}\left[\bar{G}^{-1}Y^{-3}|X=x\right]\left(K^2(1) - \int_0^1 (K^2)'(s)\chi_x(s)\mathrm{d}s\right) + o(1).$$

Hence,

$$\mathrm{Cov}(\widetilde{g_1}(x), \widetilde{g_2}(x)) = \frac{\mathrm{E}\left[\bar{G}^{-1}Y^{-3}|X=x\right]\left(K^2(1) - \int_0^1 (K^2)'(s)\chi_x(s)\mathrm{d}s\right)}{n\phi_x(h)\left(K(1) - \int_0^1 K'(s)\chi_x(s)\mathrm{d}s\right)^2} + o\left(\frac{1}{n\phi_x(h)}\right).$$

It follow that

$$\mathrm{Var}\left(\sum_{i=1}^n L_i(x)\right) = g_2^2(x)\sigma + o(1).$$

Therefore, it is sufficient to demonstrate that the numerator of (11) converges to 0 to finish the evidence of this lemma. For that we apply the C_r inequality (see Loeve (1963), p. 155) showing that

$$\sum_{i=1}^n \mathrm{E}\left[|L_i(x) - \mathrm{E}\left[L_i(x)\right]|^{2+\delta}\right] \leq C\sum_{i=1}^n \mathrm{E}\left[|L_i(x)|^{2+\delta}\right] + C'\sum_{i=1}^n |\mathrm{E}\left[L_i(x)\right]|^{2+\delta}.$$

Then, under assumptions (H5) and (H3), we get

$$\sum_{i=1}^n \mathrm{E}\left[|L_i(x)|^{2+\delta}\right] = n^{\frac{-\delta}{2}}(\phi_x(h))^{-1-\frac{\delta}{2}}\mathrm{E}\left[\delta_1^{2+\delta}\bar{G}^{-(2+\delta)}(T_1)K_1^{2+\delta}(x)|g_1(x)T_i^{-2} - g_2(x)T_i^{-1}|^{2+\delta}\right]$$
$$\leq C(n\phi_x(h))^{-1-\frac{\delta}{2}}\left(\mathrm{E}[K_1^{2+\delta}]\right) \to 0.$$

For the second term, we obtain

$$\sum_{i=1}^n |\mathrm{E}\left[L_i(x)\right]|^{2+\delta} \leq n^{\frac{-\delta}{2}}(\phi_x(h))^{-1-\frac{\delta}{2}}\left|\mathrm{E}\left[\delta_1\bar{G}^{-1}(T_1)K_1(x)|g_1(x)T_i^{-2} - g_2(x)T_i^{-1}|\right]\right|^{2+\delta}$$
$$\leq Cn^{\frac{-\delta}{2}}(\phi_x(h))^{\frac{1+\delta}{2}} \to 0$$

which finishes the proof.

Proof of Lemma 3.8. For the first term, by taking into account Lemmas 3.2-3.4 and equation (12), we have

$$\mathrm{E}\left[\widetilde{g_2}(x) - g_2(x)\right] \to 0$$

and

$$\mathrm{Var}\left[\widetilde{g_2}(x)\right] \to 0.$$

Then,

$$\widehat{g}_{2,n}(x) - g_2(x) \to 0,$$

in probability. For the second limit, by lemma 3.4 and first limit, we get

$$\mathrm{Var}\left[\widehat{g}_{l,n}(x) - \widetilde{g}_l(x)\right] \to 0.$$

Thus, it follow that

$$\left(\frac{n\phi_x(h)}{g_2^2(x)\sigma^2(x)}\right)^{\frac{1}{2}} \left[\left(\widehat{g}_{1,n}(x) - \widetilde{g}_1(x)\right) g_2(x) + \left(\widetilde{g}_2(x) - \widehat{g}_{2,n}(x)\right) g_1(x)\right] \to 0,$$

in probability.

Proof of Lemma 3.9. We write

$$\begin{aligned}
&\left[\frac{1}{g_2(x)}\left(\mathrm{E}\left[\widetilde{g}_1(x)\right] - g_1(x)\right) g_2(x) + \left(g_2(x) - \mathrm{E}\left[\widetilde{g}_2(x)\right]\right) g_1(x)\right] \\
&= \frac{1}{g_2(x)}\left[\mathrm{E}\left[\widetilde{g}_1(x)\right] g_2(x) - g_1(x)g_2(x) + g_1(x)g_2(x) - \mathrm{E}\left[\widetilde{g}_2(x)\right] g_1(x)\right] \\
&= \frac{1}{g_2(x)\mathrm{E}\left[\widetilde{g}_2(x)\right]}\left[\mathrm{E}\left[\widetilde{g}_1(x)\right] g_2(x) - \mathrm{E}\left[\widetilde{g}_2(x)\right] g_1(x)\right] \mathrm{E}\left[\widetilde{g}_2(x)\right] \\
&= A_n \mathrm{E}\left[\widetilde{g}_2(x)\right].
\end{aligned}$$

For A_n, we get

$$A_n = \frac{\mathrm{E}\left[\widetilde{g}_1(x)\right]}{\mathrm{E}\left[\widetilde{g}_2(x)\right]} - \frac{g_1(x)}{g_2(x)},$$

for which suffices to evaluate $\mathrm{E}\left[\widetilde{g}_1(x)\right]$ and $\mathrm{E}\left[\widetilde{g}_2(x)\right]$. By the same arguments used in Lemma 3.2, we obtain

$$\mathrm{E}\left[\widetilde{g}_1(x)\right] = \frac{1}{\mathrm{E}\left[K_1\right]}\mathrm{E}\left[K_1(x)\mathrm{E}\left[Y_1^{-1}|X_1\right]\right]$$

and

$$\mathrm{E}\left[\widetilde{g}_2(x)\right] = \frac{1}{\mathrm{E}\left[K_1\right]}\mathrm{E}\left[K_1(x)\mathrm{E}\left[Y_1^{-2}|X_1\right]\right].$$

By the same ideas used by Ferraty et al. (2007) for regression operator, we demonstrate that

$$\mathrm{E}\left[\widetilde{g}_1(x)\right] = g_1(x) + h\Psi_1^{'}(0)\left[\frac{K(1) - \int_0^1 (sK(s))^{'}\chi_x(s)\mathrm{d}s}{K(1) - \int_0^1 (K)^{'}(s)\chi_x(s)\mathrm{d}s}\right] + o(h)$$

and

$$\mathrm{E}\left[\widetilde{g}_2(x)\right] = g_2(x) + h\Psi_2^{'}(0)\left[\frac{K(1) - \int_0^1 (sK(s))' \chi_x(s)\mathrm{d}s}{K(1) - \int_0^1 (K)'(s)\chi_x(s)\mathrm{d}s}\right] + o(h).$$

Thus,

$$A_n = \frac{\mathrm{E}\left[\widetilde{g}_1(x)\right]}{\mathrm{E}\left[\widetilde{g}_2(x)\right]} - r(x) = hB_n(x) + o(h),$$

where

$$B_n = \frac{(\Psi_1^{'}(0) - r(x)\Psi_2^{'}(0))M_0}{M_1 g_2(x)}.$$

For the second term, we have

$$\mathrm{E}\left[\widetilde{g}_2(x)\right] = g_2(x) + h\Psi_2^{'}(0)\left[\frac{K(1) - \int_0^1 (sK(s))' \chi_x(s)\mathrm{d}s}{K(1) - \int_0^1 (K)'(s)\chi_x(s)\mathrm{d}s}\right] + o(h).$$

Then,

$$\mathrm{E}\left[\widetilde{g}_2(x)\right] - g_2(x) = O(h).$$

Hence, to show that Lemma 3.9 converges to 0 in probability, we have

$$\mathrm{E}\left[\left(\frac{n\phi_x(h)}{g_2^2(x)\sigma^2(x)}\right)^{\frac{1}{2}} A_n\left(|g_2(x) - \mathrm{E}\left[\widetilde{g}_2(x)\right]|\right)\right] = 0$$

and

$$\mathrm{Var}\left[\left(\frac{n\phi_x(h)}{g_2^2(x)\sigma^2(x)}\right)^{\frac{1}{2}} A_n\left(|g_2(x) - \mathrm{E}\left[\widetilde{g}_2(x)\right]|\right)\right] = O(A_n^2) = O(h^2) \to 0,$$

which complete the proof.

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