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Estimation for a family of skew scale-mixture distributions

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Abstract

A family of scale-mixture of epsilon Birnbaum-Saunders (SEBS) distributions is introduced and studied. This new family is based on the epsilon-skew-normal (ESN) distribution and provides more flexibility in terms of skewness and kurtosis (heavy tails). We discuss some of its probabilistic and inferential properties. We perform maximum likelihood estimation by the use of EM algorithm and then evaluate the performance of the estimators through a Monte Carlo simulation study. A bias-reduction method is suggested for reducing the bias of some estimators. Finally, the analysis of two data sets is performed for illustrative purposes.

Keywords: Birnbaum-Saunders distribution · Epsilon-skew-normal distribution · Maximum likelihood · Order statistics · Monte Carlo simulation.

Mathematics Subject Classification: Primary 65C10 · Secondary 60E05.

1. Introduction

The epsilon-skew-normal (ESN) family of distributions was introduced by Mudholkar and Hutson (2000) by splicing two half-normal distributions with different scales. The ESN family provides more flexible models, in terms of skewness, as compared with the normal model. We say that a random variable (RV) X follows a standard ESN distribution with asymmetry parameter $|\varepsilon| < 1$, denoted by $X \sim \mathrm{ESN}(\varepsilon)$, if its probability density function (PDF) is given by

$$f_{\mathrm{ESN}}(x;\epsilon) = \phi\left(\frac{x}{1+\varepsilon}\right)\mathrm{I}_{\{x<0\}} + \phi\left(\frac{x}{1-\varepsilon}\right)\mathrm{I}_{\{x\geq 0\}}, \tag{1}$$

where $\phi(\cdot)$ is the standard normal PDF and $\mathrm{I}_{\{\mathrm{A}\}}$ is an indicator function of a set A. The limits of (1) as $\varepsilon \to \pm 1$ are the half-normal distributions. The standard normal distribution is obtained when $\varepsilon = 0$. In terms of stochastic representation, we have

$$X = (1-U_\varepsilon)(1-\varepsilon)|N_1| - U_\varepsilon(1+\varepsilon)|N_2|,$$

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with U_ε, N_1 and N_2 being independent and

$$P(U_\varepsilon = 1) = (1+\varepsilon)/2 = 1 - P(U_\varepsilon = 0),$$

where N_1 and N_2 are standard normal variables; see Mudholkar and Hutson (2000). A parametric extension of the ESN family which allows the construction of bimodal asymmetric distributions has been studied by Arellano-Valle et al. (2010). Arellano-Valle et al. (2005) generalized the model in (1) by considering a symmetric density Fang et al. (1990) $f_{\mathrm{S}}(\cdot)$ instead of $\phi(\cdot)$, thus yielding the epsilon skew symmetric family of distributions. Recently, Castro et al. (2012) used this family Arellano-Valle et al. (2005) to extend the half-normal distribution.

A widely studied and used distribution is the two-parameter Birnbaum-Saunders (BS) distribution Birnbaum and Saunders (1969), which is positively skewed with nonnegative support and is related to the normal distribution by means of the stochastic representation

$$T = \frac{\beta}{4}\left[\alpha Z + \sqrt{(\alpha Z)^2 + 4}\right]^2, \tag{2}$$

where $Z \sim \mathrm{N}(0,1)$ and T is BS distributed with the notation $T \sim \mathrm{BS}(\alpha,\beta)$. The PDF of T is given by

$$f_{\mathrm{BS}}(t;\alpha,\beta) = \phi(a(t))A(t), \quad t > 0,$$

where

$$a(t) = a(t;\alpha,\beta) = \frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right), \quad A(t) = \frac{\mathrm{d}}{\mathrm{d}t}a(t) = \frac{t^{-3/2}[t+\beta]}{2\alpha\beta^{1/2}}, \tag{3}$$

and $\alpha > 0$ and $\beta > 0$ are the shape and scale parameters, respectively. The BS distribution was originally used to describe the failure time due to fatigue under cyclic loading when some kind of accumulating damnification exceeds a threshold. However, this distribution has been widely studied and applied in many fields, for example, Rieck and Nedelman (1991) proposed a log-linear model based on the BS distribution; Bhatti (2010) studied the BS autoregressive conditional duration model; Balakrishnan et al. (2011) studied some mixture models based on BS the distribution; Paula et al. (2012) proposed a robust statistical modeling using a BS-t regression model; and Leiva et al. (2014) studied autoregressive conditional duration models based on scale mixture BS (SBS) distributions proposed by Balakrishnan et al. (2009).

The scale-mixtures of normal (SMN) distributions Andrews and Mallows (1974) is a prominent class which provides flexible heavy-tailed distributions and is usually used to develop robust inference for symmetrical data. It includes the Student-t, slash and contaminated normal distributions as special cases. Branco and Dey (2001) proposed a general class of multivariate skew-elliptical distributions which contains the multivariate normal, Student's t, exponential power, and Pearson type II distributions as special cases but with an extra parameter to regulate skewness. Labra et al. (2012) discussed an extension of some standard likelihood based procedures to heteroscedastic nonlinear regression models under scale mixtures of skew-normal (SMSN) distributions. Zeller et al. (2011) studied robust estimation and local influence for linear regression models with scale mixtures of multivariate skew-normal distributions. Contreras-Reyes and Arellano-Valle (2013) introduced a robust and flexible statistical model of the age-length relationship of cardinalfish (*Epigonus crassicaudus*) based on a non-linear regression model in which the error distribution allows for heteroskedasticity and belongs to the scale mixtures of skew-normal (SMSN) distributions.

Recently, Castillo et al. (2011) introduced an extension of the BS distribution based on the epsilon skew symmetric family of distributions discussed by Arellano-Valle et al. (2005), which includes the generalized BS (GBS) distributions discussed by Vilca and Leiva (2006) as a special case when $\varepsilon = 0$.

In this paper, we take the ESN, BS and SMN models to introduce scale-mixture epsilon Birnbaum-Saunders (SEBS) models. The main aim here is to develop a statistical methodology based on the SEBS model. This methodology includes model formulation, estimation and inference for its parameters based on the maximum likelihood (ML) estimation method tackled via the expectation-maximization (EM) algorithm; see Dempster et al. (1977). Furthermore, we propose a bias correction technique for the shape parameter and apply it to the ML estimators. We evaluate the performance of the proposed methodology by Monte Carlo (MC) simulations and illustrate it with two real data sets. The approach developed here results in a class of distributions that provides (P1) higher flexibility in terms of skewness and kurtosis (heavy tails), (P2) robust estimation of parameters such as the SMN class, (P3) computation of the ML estimates of the model parameters by using the EM algorithm, and (P4) a generalization of the SBS distributions proposed by Balakrishnan et al. (2009).

The rest of the paper proceeds as follows. In Section 2, we introduce the family of SEBS distributions, discuss some properties such as moments and stochastic representations, and present some special cases of SEBS distributions. In Section 3, we discuss the ML estimation of the parameters of SEBS models by using the EM algorithm. Numerical results from a simulation study as well as an analysis of real data are presented and discussed in Section 4. Finally, in Section 5, we present some concluding remarks.

2. Scale-mixture epsilon Birnbaum-Saunders distributions

A random variable (RV) Y follows a SMN distribution with location and scale parameters $\mu \in \mathrm{R}$ and $\sigma^2 > 0$, respectively, denoted by $Y \sim \mathrm{SMN}(\mu, \sigma^2, H)$, if its PDF is given by

$$f_{\mathrm{SMN}}(y; \mu, \sigma^2, H) = \int_0^\infty \phi(y; \mu, \kappa(u)\sigma^2)\, \mathrm{d}H(u; \boldsymbol{\vartheta}), \quad y \in \mathrm{R}, \tag{4}$$

where $\phi(\cdot; \mu, \kappa(\cdot)\sigma^2)$ is the PDF of the normal distribution with mean and variance μ and $\kappa(\cdot)\sigma^2$, respectively, and $H(\cdot; \boldsymbol{\vartheta})$ is the cumulative distribution function (CDF) of a positive RV variable U and is indexed by the parameter vector $\boldsymbol{\vartheta}$. In terms of stochastic representation, we have

$$Y = \mu + \sqrt{\kappa(U)}Z, \tag{5}$$

where $Z \sim \mathrm{N}(0, \sigma^2)$ independently of U, and $\kappa(\cdot)$ is a strictly positive function. When $\mu = 0$ and $\sigma^2 = 1$, the simplified notation $Y \sim \mathrm{SMN}(H)$ will be used.

If we assume that Z in (5) has a ESN(ε) distribution with PDF in (1), then we obtain a new class of scale-mixture of epsilon-skew-normal (SMEN) distributions, denoted by $Y \sim \mathrm{SMEN}(H)$, with $\mu = 0$ and $\sigma^2 = 1$. Particular cases of the SMEN distributions include the epsilon-Student-t, epsilon-slash and epsilon-contaminated-normal distributions. Furthermore, if we consider (2) such that

$$T = \frac{\beta}{4}\left[\alpha\sqrt{\kappa(U)}Z + \sqrt{(\alpha\sqrt{\kappa(U)}Z)^2 + 4}\right]^2, \tag{6}$$

where $Y = \sqrt{\kappa(U)}Z \sim \text{SMEN}(H)$, with $Z \sim \text{ESN}(\varepsilon)$, then the RV T has a scale-mixture epsilon Birnbaum-Saunders distribution, denoted by $T \sim \text{SEBS}(\alpha, \beta, \varepsilon, H)$.

A random variable T has a SEBS distribution if its PDF is given by

$$f_{\text{SEBS}}(t; \alpha, \beta, \varepsilon, H) = \left\{ f_{\text{SMN}}\left(\frac{a(t)}{1+\varepsilon}\right) \mathrm{I}_{\{t<\beta\}} + f_{\text{SMN}}\left(\frac{a(t)}{1-\varepsilon}\right) \mathrm{I}_{\{t \geq \beta\}} \right\} A(t), \quad t > 0, \tag{7}$$

where $f_{\text{SMN}}(\cdot)$ is the SMN PDF in (4), with $\mu = 0$ and $\sigma^2 = 1$, and $a(t)$ and $A(t)$ are as given in (3). Let $T \sim \text{SEBS}(\alpha, \beta, \varepsilon, H)$. Then, the failure rate of T is

$$r_{\text{SEBS}}(t; \alpha, \beta, \varepsilon, H) = \frac{\left\{ f_{\text{SMN}}\left(\frac{a(t)}{1+\varepsilon}\right) \mathrm{I}_{\{t<\beta\}} + f_{\text{SMN}}\left(\frac{a(t)}{1-\varepsilon}\right) \mathrm{I}_{\{t \geq \beta\}} \right\} A(t)}{F_{\text{SMN}}\left(-\frac{a(t)}{1+\varepsilon}\right) \mathrm{I}_{\{t<\beta\}} + F_{\text{SMN}}\left(-\frac{a(t)}{1-\varepsilon}\right) \mathrm{I}_{\{t \geq \beta\}}}, \; t > 0,$$

where $a(t)$ and $A(t)$ are as given in (3) and $F_{\text{SMN}}(\cdot)$ is the CDF of the SMN model.

2.1 Properties of SEBS distributions

We now provide some useful properties of SEBS distributions.

Proposition 2.1 Let $T \sim \text{SEBS}(\alpha, \beta, \varepsilon, H)$. Then,

(a) $cT \sim \text{SEBS}(\alpha, c\beta, \varepsilon, H)$, $c > 0$;
(b) $T^{-1} \sim \text{SEBS}(\alpha, \beta^{-1}, \varepsilon, H)$;
(c) If $f_{\text{SEBS}}(t; \alpha, \beta, \varepsilon = 0, H)$, then $T \sim \text{SBS}(t; \alpha, \beta, H)$ Balakrishnan et al. (2009);
(d) $\lim_{\varepsilon \to 1} f_{\text{SEBS}}(t; \alpha, \beta, \varepsilon, H) = \left\{ f_{\text{SMN}}\left(\frac{a(t)}{2}\right) \mathrm{I}_{\{0<t<\beta\}} \right\} A(t)$;
(e) $\lim_{\varepsilon \to -1} f_{\text{SEBS}}(t; \alpha, \beta, \varepsilon, H) = \left\{ f_{\text{SMN}}\left(\frac{a(t)}{2}\right) \mathrm{I}_{\{t \geq \beta\}} \right\} A(t)$.

Proof. The properties in (a), (b), (c), (d) and (e) and directly obtained by the change-of-variable method and from the definition of the model.

Proposition 2.2 Let $T \sim SEBS(\alpha, \beta, \varepsilon, H)$. Then, the random variable T, given $U = u$, denoted by $T|(U = u)$, follows the EBS distribution Castillo et al. (2011) with parameters $\alpha\sqrt{\kappa(u)}$, β and ε, that is, $T|(U = u) \sim \text{EBS}(\alpha\sqrt{\kappa(u)}, \beta, \varepsilon)$.

Proposition 2.3 Let $T \sim \text{SEBS}(\alpha, \beta, \varepsilon, H)$. If the random variable $\kappa(U)$ in (5) has finite moments of all order, then the rth moment of T is given by

$$E[T^r] = \beta^r \sum_{i=0}^{r} \binom{2r}{2i} \sum_{j=0}^{i} \binom{i}{j} \left(\frac{\alpha}{2}\right)^{2(r+j-i)} E[\kappa(U)^r], \quad r = 1, 2, \ldots.$$

Proof. This result is obtained from the stochastic representation in (6) and by repeatedly using the binomial theorem.

Proposition 2.4 Let $T \sim SEBS(\alpha, \beta, \varepsilon, H)$. Then:

(a) The random variable, U, given $T = t$, that is, $U|(T = t)$, has its PDF as

$$h_{U|T}(u|t) = \frac{f_{\text{EBS}}(t; \alpha\sqrt{\kappa(u)}, \beta, \varepsilon) h(u; \boldsymbol{\vartheta})}{f_{\text{SEBS}}(t; \alpha, \beta, \varepsilon, H)};$$

(b) The moments of the random variable $\kappa(U)|(T=t)$ are given by

$$E[\{\kappa(U)\}^r|(T=t)] = \int_0^\infty [\kappa(u)]^r \frac{f_{\rm EBS}(t;\alpha\sqrt{\kappa(u)},\beta,\varepsilon)}{f_{\rm SEBS}(t;\alpha,\beta,\varepsilon,H)} {\rm d}H(u;\boldsymbol{\vartheta}), \quad r \in {\rm R}.$$

2.2 Examples of SEBS distributions

2.2.1 Epsilon Student-t-Birnbaum-Saunders distribution (EtBS)

Let $U \sim {\rm Gamma}(\nu/2,\nu/2)$, with $\nu>0$ degrees of freedom, and $\kappa(U)=1/U$. Then, it is possible to obtain the epsilon Student-t (Et) distribution, namely, $Y=\sqrt{\kappa(U)}Z \sim {\rm E}t(\nu,\varepsilon)$, with $Z \sim {\rm ESN}(\varepsilon)$. Since $Y \sim {\rm E}t(\nu,\varepsilon)$, its PDF is given by

$$f_{{\rm E}t}(y;\nu,\varepsilon) = \varsigma(\nu)\left\{\left(1+\frac{y^2}{\nu(1+\varepsilon)^2}\right)^{-\frac{\nu+1}{2}} {\rm I}_{\{y<0\}} + \left(1+\frac{y^2}{\nu(1-\varepsilon)^2}\right)^{-\frac{\nu+1}{2}} {\rm I}_{\{y\geq 0\}}\right\}, \tag{8}$$

where $\varsigma(\nu)=\frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\Gamma(\nu/2)}$, $\nu>0$ and $-1<\varepsilon<1$. From (7) and (8), we can write the PDF of the RV $T \sim {\rm E}t{\rm BS}(\alpha,\beta,\nu,\varepsilon)$ as

$$\begin{aligned} f_{{\rm E}t{\rm BS}}(t;\alpha,\beta,\nu,\varepsilon) = \varsigma(\nu)&\left\{\left(1+\frac{1}{\nu\alpha^2(1+\varepsilon)^2}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]\right)^{-\frac{\nu+1}{2}} {\rm I}_{\{t<\beta\}}\right. \\ &\left. + \left(1+\frac{1}{\nu\alpha^2(1-\varepsilon)^2}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]\right)^{-\frac{\nu+1}{2}} {\rm I}_{\{t\geq\beta\}}\right\}\frac{t^{-3/2}[t+\beta]}{2\alpha\,\beta^{1/2}}, \end{aligned}$$

with $\alpha>0$, $\beta>0$, $\nu>0$ and $-1<\varepsilon<1$. Note that if $T \sim {\rm E}t{\rm BS}(\alpha,\beta,\nu,\varepsilon)$, we have

$$E[U|T=t] = \frac{\nu+1}{\nu+\frac{a^2(t)}{(1+\varepsilon)^2}}{\rm I}_{\{t<\beta\}} + \frac{\nu+1}{\nu+\frac{a^2(t)}{(1-\varepsilon)^2}}{\rm I}_{\{t\geq\beta\}}.$$

2.2.2 Epsilon contaminated-normal-Birnbaum-Saunders distribution (ECNBS)

Let $\kappa(U)=1/U$ with U have the PDF $h_U=\nu{\rm I}_{\{u=\gamma\}}+(1-\nu){\rm I}_{\{u=1\}}$. We then obtain the epsilon contaminated-normal (ECN) distribution as $Y=\sqrt{\kappa(U)}Z \sim {\rm ECN}(\nu,\gamma,\varepsilon)$, with $Z \sim {\rm ESN}(\varepsilon)$, and its PDF given by

$$\begin{aligned} f_{\rm ECN}(y;\nu,\gamma,\varepsilon) = &\left\{\nu\phi\left(\frac{y}{1+\varepsilon};0,\frac{1}{\gamma}\right)+(1-\nu)\phi\left(\frac{y}{1+\varepsilon};0,1\right)\right\}{\rm I}_{\{y<0\}} \\ &+\left\{\nu\phi\left(\frac{y}{1-\varepsilon};0,\frac{1}{\gamma}\right)+(1-\nu)\phi\left(\frac{y}{1-\varepsilon};0,1\right)\right\}{\rm I}_{\{y\geq0\}}. \end{aligned} \tag{9}$$

From (7) and (9), we obtain the PDF of the RV $T \sim {\rm ECNBS}(\alpha,\beta,\nu,\gamma,\varepsilon)$ as

$$\begin{aligned} f_{\rm ECNBS}(t;\alpha,\beta,\nu,\gamma,\varepsilon) = &\left\{\left[\nu\phi\left(\frac{a(t)}{1+\varepsilon};0,\frac{1}{\gamma}\right)+(1-\nu)\phi\left(\frac{a(t)}{1+\varepsilon};0,1\right)\right]{\rm I}_{\{t<\beta\}}\right. \\ &\left.+\left[\nu\phi\left(\frac{a(t)}{1-\varepsilon};0,\frac{1}{\gamma}\right)+(1-\nu)\phi\left(\frac{a(t)}{1-\varepsilon};0,1\right)\right]{\rm I}_{\{t\geq\beta\}}\right\}\frac{t^{-3/2}[t+\beta]}{2\alpha\,\beta^{1/2}}, \end{aligned}$$

with $\alpha > 0$, $\beta > 0$, $0 < \nu < 1$, $0 < \gamma < 1$ and $-1 < \varepsilon < 1$, where $a(t)$ is as in (3). If $T \sim \mathrm{ECNBS}(\alpha, \beta, \nu, \gamma, \varepsilon)$, then

$$\begin{aligned}
E[U|T=t] = {} & \frac{1-\nu+\nu\gamma^{3/2}\exp\left(\frac{(1-\gamma)a^2(t)}{2(1+\varepsilon)^2}\right)}{1-\nu+\nu\gamma^{1/2}\exp\left(\frac{(1-\gamma)a^2(t)}{2(1+\varepsilon)^2}\right)}\mathrm{I}_{\{t<\beta\}} \\
& + \frac{1-\nu+\nu\gamma^{3/2}\exp\left(\frac{(1-\gamma)a^2(t)}{2(1-\varepsilon)^2}\right)}{1-\nu+\nu\gamma^{1/2}\exp\left(\frac{(1-\gamma)a^2(t)}{2(1-\varepsilon)^2}\right)}\mathrm{I}_{\{t\geq\beta\}}.
\end{aligned}$$

2.2.3 Epsilon slash-Birnbaum-Saunders distribution (ESLBS)

Now, let $\kappa(U) = 1/U$ and U be a $\mathrm{Beta}(\nu, 1)$ distributed RV, with $0 < u < 1$ and $\nu > 0$. Then, it follows that a RV Y following an epsilon slash (ESL) distribution, that is, $Y = \sqrt{\kappa(U)}Z \sim \mathrm{ESL}(\nu, \varepsilon)$, with $Z \sim \mathrm{ESN}(\varepsilon)$, has its PDF as

$$\begin{aligned}
f_{\mathrm{ESL}}(y;\nu,\varepsilon) = {} & \left\{\nu\int_0^1 u^{\nu-1}\phi\left(\frac{y}{1+\varepsilon};0,\frac{1}{u}\right)\mathrm{d}u\right\}\mathrm{I}_{\{y<0\}} \\
& + \left\{\nu\int_0^1 u^{\nu-1}\phi\left(\frac{y}{1-\varepsilon};0,\frac{1}{u}\right)\mathrm{d}u\right\}\mathrm{I}_{\{y\geq 0\}},
\end{aligned} \tag{10}$$

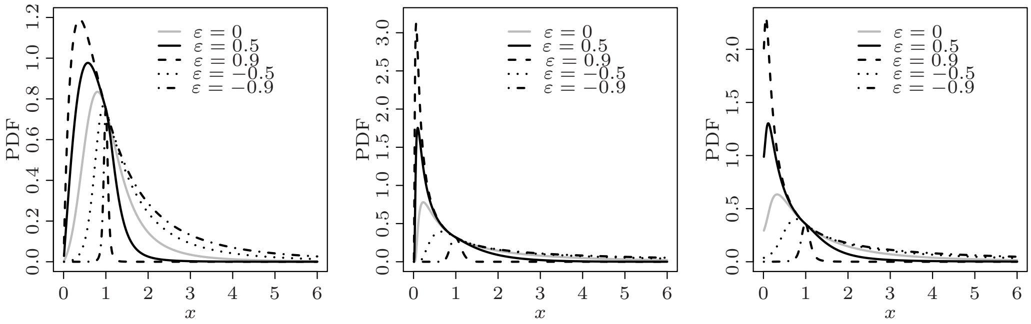
From (7) and (10), the PDF of the RV $T \sim \mathrm{ESLBS}(\alpha, \beta, \nu, \varepsilon)$ is obtained as

$$\begin{aligned}
f_{\mathrm{ESLBS}}(t;\alpha,\beta,\nu,\varepsilon) = \nu & \left\{\left[\int_0^1 u^{\nu-1}\phi\left(\frac{a(t)}{1+\varepsilon};0,\frac{1}{u}\right)\mathrm{d}u\right]\mathrm{I}_{\{t<\beta\}}\right. \\
& \left. + \left[\int_0^1 u^{\nu-1}\phi\left(\frac{a(t)}{1-\varepsilon};0,\frac{1}{u}\right)\mathrm{d}u\right]\mathrm{I}_{\{t\geq\beta\}}\right\}\frac{t^{-3/2}[t+\beta]}{2\alpha\,\beta^{1/2}},
\end{aligned}$$

with $\alpha > 0$, $\beta > 0$, $\nu > 0$ and $-1 < \varepsilon < 1$, where $a(t)$ is as in (3). Note that if $T \sim \mathrm{ESLBS}(\alpha, \beta, \nu, \varepsilon)$, we have

$$\begin{aligned}
E[U|T=t] = {} & \frac{(2\nu+1)}{a^2(t)/(1+\varepsilon)^2}\,\frac{P_1\left(\nu+\frac{3}{2},\frac{a^2(t)}{2(1+\varepsilon)^2}\right)}{P_1\left(\nu+\frac{1}{2},\frac{a^2(t)}{2(1+\varepsilon)^2}\right)}\mathrm{I}_{\{t<\beta\}} \\
& + \frac{(2\nu+1)}{a^2(t)/(1-\varepsilon)^2}\,\frac{P_1\left(\nu+\frac{3}{2},\frac{a^2(t)}{2(1-\varepsilon)^2}\right)}{P_1\left(\nu+\frac{1}{2},\frac{a^2(t)}{2(1-\varepsilon)^2}\right)}\mathrm{I}_{\{t\geq\beta\}},
\end{aligned}$$

where $P_x(m, n)$ is the CDF of the gamma distribution at x. The PDFs of EtBS, ECNBS and ESLBS distributions are displayed in Figure 1 for several choices of skewness (ε) parameter values. These PDFs are all positively skewed and unimodal. Note that the skewness parameter permits greater flexibility. Note also that if $\varepsilon = 0$, we get the tBS, CNBS and SLBS distributions; see Díaz-García and Leiva (2005) and Balakrishnan et al. (2009).



(a) EtBS($\alpha = 0.5, \beta = 1, \nu = 4, \varepsilon$) (b) ECNBS($\alpha = 1.25, \beta = 1, \nu = 3, \gamma = 1, \varepsilon$) (c) ESLBS($\alpha = 0.75, \beta = 1, \nu = 1, \varepsilon$)

Figure 1. Plots of PDFs of various distributions for different values of parameters.

3. Estimation procedures

3.1 Maximum likelihood estimation

Let $t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(n)}$ denote the order statistics from a random sample $T_1, T_2, \ldots, T_n$ from the SEBS$(\alpha, \beta, \varepsilon, H)$ distribution. Considering $\boldsymbol{\vartheta}$ known, let $\tau \equiv \tau(t_{(1)}, t_{(2)}, \ldots, t_{(n)}, \beta)$, where $t_{(0)} = 0$ and $t_{(n+1)} = \infty$, be such that $t_{(\tau)} < \beta < t_{(\tau+1)}$. The log-likelihood function can be expressed as

$$\ell(\boldsymbol{\theta}) \propto -n\log(\alpha) - \frac{n}{2}\log(\beta) + \sum_{i=1}^{n}\log(t_{(i)} + \beta) + \sum_{i=1}^{\tau}\log\left(f_{\rm SMN}\left(\frac{a(t_{(i)})}{1+\varepsilon}\right)\right)$$
$$+ \sum_{i=\tau+1}^{n}\log\left(f_{\rm SMN}\left(\frac{a(t_{(i)})}{1-\varepsilon}\right)\right), \tag{11}$$

where $\boldsymbol{\theta} = (\alpha, \beta, \varepsilon)^\top$ and $a(t_{(i)}) = \frac{1}{\alpha}\left(\sqrt{\frac{t_{(i)}}{\beta}} - \sqrt{\frac{\beta}{t_{(i)}}}\right)$. The ML estimate is obtained as the value of $\widehat{\boldsymbol{\theta}}_\tau = \arg\max \ell(\widehat{\boldsymbol{\theta}}_\tau)$, for $\tau = 0, 1, \ldots, n$. Note that if $\tau = 0$, the log-likelihood in (11) becomes

$$\ell(\boldsymbol{\theta}) \propto -n\log(\alpha) - \frac{n}{2}\log(\beta) + \sum_{i=1}^{n}\log(t_{(i)} + \beta) + \sum_{i=1}^{n}\log\left(f_{\rm SMN}\left(\frac{1}{\alpha(1-\varepsilon)}\left[\sqrt{\frac{t_{(i)}}{\beta}} - \sqrt{\frac{\beta}{t_{(i)}}}\right]\right)\right),$$

and the maximum, in terms of ε, occurs at $\widehat{\varepsilon}_0 = -1$. Therefore, $\beta < t_{(1)}$, which implies that $\widehat{\beta}_0 = t_{(1)}$, and $\widehat{\alpha}_0$ turns out to be the solution of

$$\frac{\partial\ell(\boldsymbol{\theta})}{\partial\alpha} = -\frac{n}{\alpha} + \sum_{i=2}^{n}\frac{\partial}{\partial\alpha}\log\left(f_{\rm SMN}\left(\frac{1}{\alpha(1-\varepsilon)}\left[\sqrt{\frac{t_{(i)}}{t_{(1)}}} - \sqrt{\frac{t_{(1)}}{t_{(i)}}}\right]\right)\right) = 0.$$

On the other hand, if $\tau = n$, the log-likelihood in (11) reduces to

$$\ell(\boldsymbol{\theta}) \propto -n\log(\alpha) - \frac{n}{2}\log(\beta) + \sum_{i=1}^{n}\log(t_{(i)} + \beta) + \sum_{i=1}^{n}\log\left(f_{\rm SMN}\left(\frac{1}{\alpha(1+\varepsilon)}\left[\sqrt{\frac{t_{(i)}}{\beta}} - \sqrt{\frac{\beta}{t_{(i)}}}\right]\right)\right).$$

Its maximum, in terms of ε, is obtained when $\widehat{\varepsilon}_n = 1$. Hence, $\beta > t_{(n)}$, which implies that $\widehat{\beta}_n = t_{(n)}$, and from this we get $\widehat{\alpha}_n$ as the solution of the equation

$$\frac{\partial\ell(\boldsymbol{\theta})}{\partial\alpha} = -\frac{n}{\alpha} + \sum_{i=1}^{n-1}\frac{\partial}{\partial\alpha}\log\left(f_{\rm SMN}\left(\frac{1}{\alpha(1+\varepsilon)}\left[\sqrt{\frac{t_{(i)}}{t_{(n)}}} - \sqrt{\frac{t_{(n)}}{t_{(i)}}}\right]\right)\right) = 0.$$

Generally, if $1 \leq \tau < n$, then the ML estimates $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\varepsilon}$ of α, β and ε are obtained from the maximization of (11) as the solution to the following system of equations:

$$\begin{aligned}
\frac{\partial \ell(\boldsymbol{\theta})}{\partial \alpha} &= -\frac{n}{\alpha} + \sum_{i=1}^{\tau} \frac{\partial}{\partial \alpha} \log\left(f_{\mathrm{SMN}}\left(\frac{a(t_{(i)})}{1+\varepsilon}\right)\right) + \sum_{i=\tau+1}^{n} \frac{\partial}{\partial \alpha} \log\left(f_{\mathrm{SMN}}\left(\frac{a(t_{(i)})}{1-\varepsilon}\right)\right) = 0, \\
\frac{\partial \ell(\boldsymbol{\theta})}{\partial \beta} &= -\frac{n}{2\beta} + \sum_{i=1}^{n} \frac{1}{t_{(i)}+\beta} + \sum_{i=1}^{\tau} \frac{\partial}{\partial \beta} \log\left(f_{\mathrm{SMN}}\left(\frac{a(t_{(i)})}{1+\varepsilon}\right)\right) \\
&\quad + \sum_{i=\tau+1}^{n} \frac{\partial}{\partial \beta} \log\left(f_{\mathrm{SMN}}\left(\frac{a(t_{(i)})}{1-\varepsilon}\right)\right) = 0, \\
\frac{\partial \ell(\boldsymbol{\theta})}{\partial \varepsilon} &= \sum_{i=1}^{\tau} \frac{\partial}{\partial \varepsilon} \log\left(f_{\mathrm{SMN}}\left(\frac{a(t_{(i)})}{1+\varepsilon}\right)\right) + \sum_{i=\tau+1}^{n} \frac{\partial}{\partial \varepsilon} \log\left(f_{\mathrm{SMN}}\left(\frac{a(t_{(i)})}{1-\varepsilon}\right)\right) = 0.
\end{aligned}$$

Proposition 3.1 The ML estimator of $\boldsymbol{\theta}$, $\widehat{\boldsymbol{\theta}}_\tau$, is obtained for a τ such that $\ell(\widehat{\boldsymbol{\theta}}_\tau) > \ell(\widehat{\boldsymbol{\theta}}_l)$, for $l \in \{0, 1, \ldots, n\}$.

3.2 Estimation through the EM algorithm

We now discuss the ML estimation of the model parameters by using the EM algorithm. Let $\boldsymbol{t} = (t_1, \ldots, t_n)$ denote the observed data and $\boldsymbol{u} = (u_1, \ldots, u_n)$ be the unobserved data, so that the complete data $\boldsymbol{t}_c = (\boldsymbol{t}^\top, \boldsymbol{u}^\top)^\top$ corresponds to the original data augmented with $\boldsymbol{u}$. Let $t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(n)}$ be the order statistics from a random sample $T_1, T_2, \ldots, T_n$, where $T_i \sim \mathrm{SEBS}(\alpha, \beta, \varepsilon)$, for $i = 1, \ldots, n$. From Proposition 2.2, it follows that

$$\begin{aligned}
T_i | U_i = u_i &\overset{\text{ind}}{\sim} \mathrm{EBS}(\alpha\sqrt{\kappa(u_i)}, \beta, \varepsilon), \\
U_i &\overset{\text{ind}}{\sim} h(u_i; \boldsymbol{\vartheta}), \quad i = 1, \ldots, n,
\end{aligned} \tag{12}$$

where $\boldsymbol{\vartheta}$ is assumed to be known. Then, the complete log-likelihood associated with $\boldsymbol{t}_c$ is given by

$$\begin{aligned}
\ell(\boldsymbol{\theta}|\boldsymbol{t}_c) \propto{}& -n\log(\alpha) - \frac{n}{2}\log(\beta) + \sum_{i=1}^{n} \log(t_{(i)} + \beta) - \frac{1}{2(1+\varepsilon)^2\alpha^2} \sum_{i=1}^{\tau} \frac{1}{\kappa(u_{(i)})}\left[\frac{t_{(i)}}{\beta} + \frac{\beta}{t_{(i)}} - 2\right] \\
& - \frac{1}{2(1-\varepsilon)^2\alpha^2} \sum_{i=\tau+1}^{n} \frac{1}{\kappa(u_{(i)})}\left[\frac{t_{(i)}}{\beta} + \frac{\beta}{t_{(i)}} - 2\right],
\end{aligned}$$

where $u_{(1)} \leq u_{(2)} \leq \cdots \leq u_{(n)}$ are the order statistics from the random sample $U_1, U_2, \ldots, U_n$, $\boldsymbol{\theta} = (\alpha, \beta, \varepsilon)^\top$, and τ is as defined in Subsection 3.1. The expected value of the complete data log-likelihood function, conditioned on the observed data, given the current estimate $\widehat{\boldsymbol{\theta}}^{(k)} = (\widehat{\alpha}^{(k)}, \widehat{\beta}^{(k)}, \widehat{\varepsilon}^{(k)})^\top$, is given by

$$
\begin{aligned}
Q(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}}^{(k)}) &= E[\ell(\boldsymbol{\theta}|\boldsymbol{t}_c)|\boldsymbol{t}, \widehat{\boldsymbol{\theta}}^{(k)}] \\
&= -n\log(\widehat{\alpha}^{(k)}) - \frac{n}{2}\log(\widehat{\beta}^{(k)}) + \sum_{i=1}^{n}\log(t_{(i)} + \widehat{\beta}^{(k)}) - \frac{1}{2(1+\widehat{\varepsilon}^{(k)})^2\widehat{\alpha}^{2(k)}} \\
&\quad \sum_{i=1}^{\tau}\widehat{\kappa}_{(i)}^{(k)}\left[\frac{t_{(i)}}{\widehat{\beta}^{(k)}} + \frac{\widehat{\beta}^{(k)}}{t_{(i)}} - 2\right] - \frac{1}{2(1-\widehat{\varepsilon}^{(k)})^2\widehat{\alpha}^{2(k)}}\sum_{i=\tau+1}^{n}\widehat{\kappa}_{(i)}^{(k)}\left[\frac{t_{(i)}}{\widehat{\beta}^{(k)}} + \frac{\widehat{\beta}^{(k)}}{t_{(i)}} - 2\right],
\end{aligned}
$$

with $\boldsymbol{\theta} = (\alpha, \beta, \varepsilon)^\top$ and $\widehat{\kappa}_{(i)}^{(k)} = E[\kappa^{-1}(U_{(i)})|t_{(i)}, \widehat{\boldsymbol{\theta}}^{(k)}]$.

For determining the ML estimates of the SEBS model parameters by using the EM algorithm, we proceed as follows:

- **E-step.** Given $\widehat{\boldsymbol{\theta}} = \widehat{\boldsymbol{\theta}}^{(k)} = (\widehat{\alpha}^{(k)}, \widehat{\beta}^{(k)}, \widehat{\varepsilon}^{(k)})^\top$, compute for $i = 1, \ldots, n$, $k = 1, 2, \ldots$, $\widehat{\kappa}_i^{(k)} = E[\kappa^{-1}(U_i)|t_i, \widehat{\boldsymbol{\theta}}^{(k)}]$;
- **M-step.** Update $\widehat{\boldsymbol{\theta}}^{(k+1)}$ by maximizing $Q(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}}^{(k)})$ over $\boldsymbol{\theta}$. In this step, we have two cases as follows:

  Case 1) If $\tau = 0$ or $\tau = n$, we have

$$
(\widehat{\alpha}^{(k+1)}, \widehat{\beta}^{(k+1)}, \widehat{\varepsilon}^{(k+1)}) = \begin{cases} (\widehat{\alpha}_0^{(k+1)}, t_{(1)}, -1) & \text{if } \tau = 0, \\ (\widehat{\alpha}_n^{(k+1)}, t_{(n)}, 1) & \text{if } \tau = n, \end{cases}
$$

  where

$$
\widehat{\alpha}_0^{(k+1)} = \left(\frac{1}{4n}\sum_{i=2}^{n}\widehat{\kappa}_{(i)}^{(k)}\left[\frac{t_{(i)}}{t_{(1)}} + \frac{t_{(1)}}{t_{(i)}} - 2\right]\right)^{1/2}
$$

  and

$$
\widehat{\alpha}_n^{(k+1)} = \left(\frac{1}{4n}\sum_{i=1}^{n-1}\widehat{\kappa}_{(i)}^{(k)}\left[\frac{t_{(i)}}{t_{(n)}} + \frac{t_{(n)}}{t_{(i)}} - 2\right]\right)^{1/2};
$$

  Case 2) If $1 \leq \tau < n$, we maximize $Q(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}}^{(k)})$ over $\boldsymbol{\theta}$, yielding the equations

$$
\begin{aligned}
\widehat{\alpha}^{2(k+1)} &= \tfrac{1}{(1+\widehat{\varepsilon}^{(k)})^2 n}\textstyle\sum_{i=1}^{\tau}\widehat{\kappa}_{(i)}^{(k)}\left[\tfrac{t_{(i)}}{\widehat{\beta}^{(k)}} + \tfrac{\widehat{\beta}^{(k)}}{t_{(i)}} - 2\right] + \tfrac{1}{(1-\widehat{\varepsilon}^{(k)})^2 n}\textstyle\sum_{i=\tau+1}^{n}\widehat{\kappa}_{(i)}^{(k)}\left[\tfrac{t_{(i)}}{\widehat{\beta}^{(k)}} + \tfrac{\widehat{\beta}^{(k)}}{t_{(i)}} - 2\right], \\
&-\tfrac{\sum_{i=1}^{\tau}\widehat{\kappa}_{(i)}^{(k)}\left(\frac{1}{t_{(i)}} - \frac{t_{(i)}}{\widehat{\beta}^{2(k)}}\right)}{2\widehat{\alpha}^{2(k)}(1+\widehat{\varepsilon}^{(k)})^2} - \tfrac{\sum_{i=\tau+1}^{n}\widehat{\kappa}_{(i)}^{(k)}\left(\frac{1}{t_{(i)}} - \frac{t_{(i)}}{\widehat{\beta}^{2(k)}}\right)}{2\widehat{\alpha}^{2(k)}(1-\widehat{\varepsilon}^{(k)})^2} + \textstyle\sum_{i=1}^{n}\tfrac{1}{\widehat{\beta}^{(k)}+t_{(i)}} - \tfrac{n}{2\widehat{\beta}^{(k)}} = 0, \\
\widehat{\varepsilon}^{(k+1)} &= \tfrac{1-\widehat{\Psi}^{(k)}}{1+\widehat{\Psi}^{(k)}},
\end{aligned}
$$

where

$$\widehat{\Psi}^{(k)} = \left(\frac{\sum_{i=\tau+1}^{n} \widehat{\kappa}_{(i)}^{(k)} \left(\frac{\widehat{\beta}^{(k)}}{t_{(i)}} + \frac{t_{(i)}}{\widehat{\beta}^{(k)}} - 2 \right)}{\sum_{i=1}^{\tau} \widehat{\kappa}_{(i)}^{(k)} \left(\frac{\widehat{\beta}^{(k)}}{t_{(i)}} + \frac{t_{(i)}}{\widehat{\beta}^{(k)}} - 2 \right)} \right)^{1/3}.$$

Proposition 3.2 The ML estimator of $\boldsymbol{\theta}$ is given by $\widehat{\boldsymbol{\theta}}_{\tau}^{(k+1)}$, where τ is such that $Q(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}}_{\tau}^{(k)}) > Q(\boldsymbol{\theta}|\widehat{\boldsymbol{\theta}}_{l}^{(k)})$, for $l \in \{0, 1, \ldots, n\}$.

In the EM algorithm, the iterations are carried out successively until a certain convergence criterion is satisfied, for instance, when $||\widehat{\boldsymbol{\theta}}^{(k+1)} - \widehat{\boldsymbol{\theta}}^{(k)}||$ is sufficiently small. As starting values $\widehat{\alpha}^{(0)}$ and $\widehat{\beta}^{(0)}$, we can use, for example, the modified moment estimates Ng et al. (2003), and $\widehat{\varepsilon}^{(0)}$, the moment estimates Mudholkar and Hutson (2000), given by

$$\widehat{\varepsilon}^{(0)} \approx -0.5835\sqrt{b_1} - 0.5861(\sqrt{b_1})^3 + 1.0763(\sqrt{b_1})^5 - 0.9226(\sqrt{b_1})^7,$$

where $\sqrt{b_1}$ is the sample coefficient of skewness.

We use the profile likelihood for finding the values of $\boldsymbol{\vartheta}$. For example, for the EtBS distribution, we have the following two steps to find an estimate of ν:

i) Let $\nu_i = i$ and for each $i = 1, \ldots, 50$ compute the ML estimates of α, β and ε by using the EM algorithm, and then compute the corresponding likelihood function;
ii) Choose the final estimate of ν as the one which maximizes the likelihood function and then select the associated estimates of α, β and ε as the final estimates.

3.3 Residual analysis

In order to assess goodness-of-fit and departures from the assumptions of the model, a residual analysis is performed. In particular, we consider the generalized Cox-Snell (GCS) residual, which is given by

$$r_i^{\text{GCS}} = -\log(\widehat{S}(x_i)), \quad i = 1, \ldots, n,$$

where $\widehat{S}$ is the survival function fitted to the data. If the model is correctly specified, then the GCS residual will have a unit exponential distribution, EXP(1) in short, whatever the model specification; see Leao et al. (2017).

4. Empirical evaluation

The ML computations through the EM algorithm for the SEBS distributions are discussed in this section. We present the results of a simulation study and then use a real data set to illustrate the proposed methodology.

4.1 Simulation study

We carried out a simulation study for evaluating the behavior of the ML estimates obtained by the EM algorithm. We consider the simpler case, that is, when $\tau \in \{0, n\}$; see Case #1 of Section 3.2. For illustrative purpose, we focus on the EtBS model, but in the next section we consider all the models. The simulation scenario assume the sample sizes $n \in$

$\{10, 50, 100\}$, values of the shape parameter as $\alpha \in \{0.10, 0.25, 0.50, 1.00, 1.25, 1.50\}$, 10,000 MC replications, and without loss of generality the value of β and ν were set as 1.00 and 4.00, respectively. Note that the values of the shape parameter α were chosen so as order to study the performance under low, moderate and high skewness.

By analyzing the results of several MC simulation studies and upon inspecting the pattern of the bias of the estimators of α, $\widehat{\alpha}_0$ and $\widehat{\alpha}_n$ (Case #1 of Section 3.2), we observed that

$$\mathrm{Bias}(\widetilde{\alpha}_0) \approx \mathrm{Bias}(\widetilde{\alpha}_n) \approx -\frac{10n}{(26n-20)}\alpha.$$

Then, by implementing a standard bias-reduction method, we propose bias-reduced ML estimators of α, denoted by $\widetilde{\alpha}_0^*$ and $\widetilde{\alpha}_n^*$, as

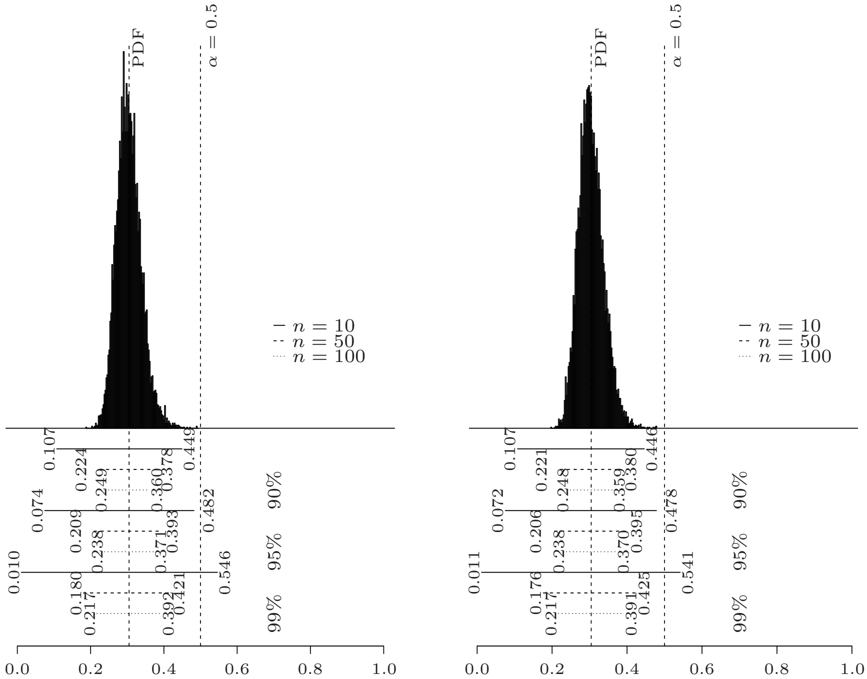
$$\widetilde{\alpha}_0^* = -\frac{(26n-20)}{(16n-20)}\widetilde{\alpha}_0 \quad \text{and} \quad \widetilde{\alpha}_n^* = -\frac{(26n-20)}{(16n-20)}\widetilde{\alpha}_n.$$

For each combination of parameters and sample size, we present the bias and mean squared error (MSE) of the ML estimates in Table 1. From this table, we note that, as the sample size increases, the ML estimates become more efficient, as expected. Also, in general, as the value of the shape parameter α increases, we observe that the performance of the estimators of β deteriorate. It is also worth noting that the bias-reduced estimators outperform the ML estimates by providing estimators with smaller biases.

In Figures 2 and 3, we show the empirical distributions of the estimators of $\alpha = 0.5$ from the MC simulation study presented in Table 1. We consider the ML estimators, $\widehat{\alpha}_0$ and $\widehat{\alpha}_n$, and their bias-reduced counterparts, $\widehat{\alpha}_0^*$ and $\widehat{\alpha}_n^*$. Similar behaviour was observed for other values of α as well. Generally, the empirical distributions of all the estimators tend to normal, as n increases. Note, however, that the mean of the empirical distributions of the bias-reduced estimators converge to the true mean. Figures 2 and 3 also provide confidence intervals (CIs) for the corresponding parameters, for several values of the sample size (n) and confidence level (ξ).

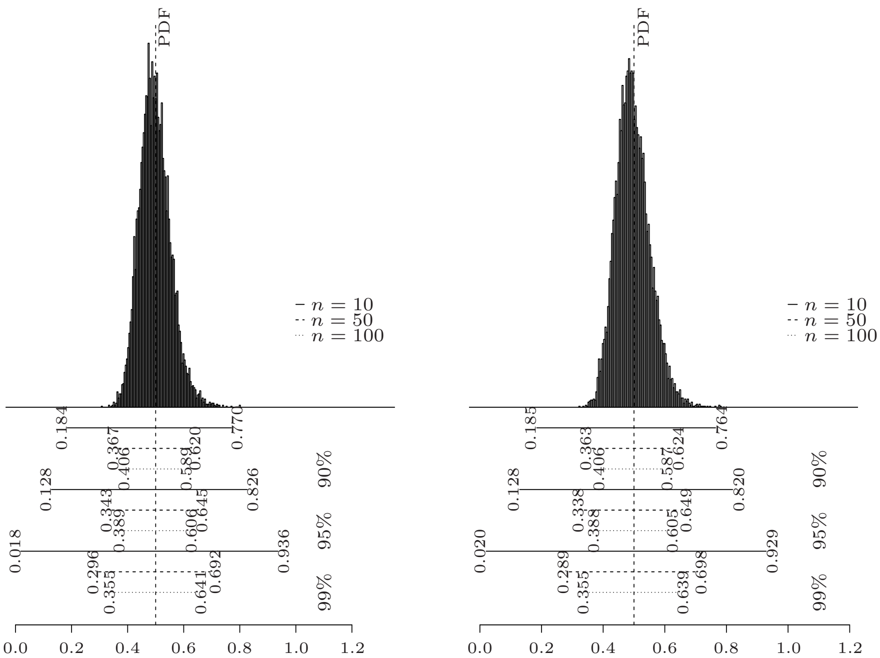
Table 1. Empirical bias and MSE (within parentheses) of the ML estimators (via the EM algorithm) for the EtBS model parameters by using the indicated sample sizes and parameter values with simulated data.

		EtBS($\beta = 1.0, \varepsilon = -1.0, \nu = 4.0$)			EtBS($\beta = 1.0, \varepsilon = 1.0, \nu = 4.0$)		
		$\tau = 0$			$\tau = n$		
n	α	$\widehat{\alpha}_0$	$\widehat{\alpha}_0^*$	$\widehat{\beta}_0$	$\widehat{\alpha}_n$	$\widehat{\alpha}_n^*$	$\widehat{\beta}_n$
10	0.10	−0.0427 (0.0024)	−0.0018 (0.0017)	0.0125 (0.0003)	−0.0425 (0.0024)	−0.0015 (0.0017)	−0.0122 (0.0003)
	0.25	−0.1078 (0.0146)	−0.0062 (0.0089)	0.0325 (0.0020)	−0.1094 (0.0150)	−0.0089 (0.0091)	−0.0295 (0.0016)
	0.50	−0.2223 (0.0606)	−0.0239 (0.0336)	0.0649 (0.0084)	−0.2233 (0.0606)	−0.0256 (0.0322)	−0.0577 (0.0059)
	1.00	−0.4714 (0.2553)	−0.0939 (0.1060)	0.1359 (0.0383)	−0.4725 (0.2574)	−0.0957 (0.1096)	−0.1114 (0.0213)
	1.25	−0.6105 (0.4183)	−0.1537 (0.1577)	0.1779 (0.0668)	−0.6079 (0.4158)	−0.1492 (0.1582)	−0.1330 (0.0298)
	1.50	−0.7523 (0.6270)	−0.2183 (0.2269)	0.2243 (0.1098)	−0.7508 (0.6228)	−0.2157 (0.2200)	−0.1558 (0.0406)
50	0.10	−0.0385 (0.0016)	0.0009 (0.0003)	0.0026 (<0.0001)	−0.0382 (0.0016)	0.0013 (0.0003)	−0.0026 (<0.0001)
	0.25	−0.0970 (0.0101)	0.0010 (0.0018)	0.0065 (0.0001)	−0.0972 (0.0101)	0.0007 (0.0017)	−0.0066 (0.0001)
	0.50	−0.1988 (0.0419)	−0.0057 (0.0063)	0.0133 (0.0004)	−0.1986 (0.0418)	−0.0055 (0.0063)	−0.0129 (0.0003)
	1.00	−0.4224 (0.1856)	−0.0522 (0.0221)	0.0270 (0.0015)	−0.4224 (0.1855)	−0.0522 (0.0218)	−0.0253 (0.0012)
	1.25	−0.5431 (0.3048)	−0.0899 (0.0348)	0.0336 (0.0023)	−0.5422 (0.3040)	−0.0885 (0.0348)	−0.0315 (0.0019)
	1.50	−0.6695 (0.4620)	−0.1371 (0.0560)	0.0403 (0.0033)	−0.6697 (0.4615)	−0.1374 (0.0539)	−0.0383 (0.0028)
100	0.10	−0.0379 (0.0015)	0.0014 (0.0001)	0.0013 (<0.0001)	−0.0377 (0.0015)	0.0017 (0.0001)	−0.0013 (<0.0001)
	0.25	−0.0953 (0.0094)	0.0029 (0.0009)	0.0033 (<0.0001)	−0.0953 (0.0094)	0.0025 (0.0009)	−0.0033 (<0.0001)
	0.50	−0.1960 (0.0396)	−0.0036 (0.0031)	0.0067 (0.0001)	−0.1955 (0.0394)	−0.0027 (0.0032)	−0.0066 (0.0001)
	1.00	−0.4136 (0.1748)	−0.0425 (0.0117)	0.0134 (0.0004)	−0.4140 (0.1752)	−0.0431 (0.0119)	−0.0129 (0.0003)
	1.25	−0.5337 (0.2898)	−0.0803 (0.0197)	0.0166 (0.0006)	−0.5321 (0.2884)	−0.0777 (0.0201)	−0.0162 (0.0005)
	1.50	−0.6577 (0.4395)	−0.1246 (0.0341)	0.0202 (0.0008)	−0.6580 (0.4398)	−0.1251 (0.0339)	−0.0197 (0.0008)



(a) $\widehat{\alpha}_0$ ($\varepsilon = -1$) (b) $\widehat{\alpha}_n$ ($\varepsilon = 1$)

Figure 2. Empirical distributions of $\widehat{\alpha}_0$ (left) and $\widehat{\alpha}_n$ (right) and their corresponding $(1-\xi) \times 100\%$ CI for the indicated n and ξ with $\alpha = 0.50$.



(a) $\widehat{\alpha}_0^*$ ($\varepsilon = -1$) (b) $\widehat{\alpha}_n^*$ ($\varepsilon = 1$)

Figure 3. Empirical distributions of $\widehat{\alpha}_0^*$ (left) and $\widehat{\alpha}_n^*$ (right) and their corresponding $(1-\xi) \times 100\%$ CI for the indicated n and ξ with $\alpha = 0.50$.

4.2 Illustrative examples

We now illustrate the proposed methodology by applying the SEBS distributions described in Section 2 to two real data sets. For comparison, the results of the BS Birnbaum and Saunders (1969) and EBS Castillo et al. (2011) distributions, are given as well.

4.2.1 Example 1

The first data set consists of fatigue lifetimes in cycles ($\times 10^{-3}$) of aluminum specimens of type 6061-T6, subject to a pressure with maximum stress of 21,000 psi (psi21). This data set was previously analyzed by Birnbaum and Saunders (1969), Balakrishnan et al. (2009) and Castillo et al. (2011). Some descriptive statistics of these data, including measures of central tendency, standard deviation (SD), coefficient of variation (CV), skewness (CS) and kurtosis (CK), are presented in Table 2.

Table 2. Summary statistics for the psi21 data.

n	Min.	Median	Mean	Max.	SD	CV	CS	CK
101	370	1416.00	1400.84	2440	391.01	27.91%	0.14	−0.28

We have given the parameter estimates for the five models in Table 3. The standard errors (SEs) were computed by using the observed information matrix given in Appendix. The values of the maximized log-likelihood and the likelihood ratio (LR) test $\text{LR} = -2(\ell(\widehat{\boldsymbol{\theta}}_{\text{BS}}) - \ell(\widehat{\boldsymbol{\theta}}_{\text{SEBS}}))$, based on the 5% critical value from the chi-square distribution with one degree of freedom ($\chi^2_1 = 3.84$), reveals that the SEBS distributions (EtBS and ECNBS) provide better adjustments compared to the other models.

Table 3. ML estimates (with SE in parentheses) determined from the EM algorithm for the indicated parameters with the psi21 data.

	Existing models		Proposed models	
Parameter	BS	EBS	EtBS	ECNBS
α	0.310	0.298	0.278	0.270
	(0.022)	(0.021)	(0.022)	(0.020)
β	1336.037	1679.145	1625.638	1631.23
	(40.749)	(48.031)	(45.088)	(42.640)
ε	–	0.483	0.413	0.417
	–	(0.056)	(0.056)	(0.052)
ν	–	–	13	0.03
γ	–	–	–	0.13
$\ell(\widehat{\boldsymbol{\theta}})$	−751.332	−746.727	−745.970	−745.368
LR	–	9.21	10.724	11.928

Figure 4 displays the QQ plots with simulated envelope of the GCS residuals for the distributions considered in Table 3. From this figure, observe that the GCS residuals show better agreement with the EXP(1) distribution distribution in the EtBS and ECNBS models.

4.2.2 Example 2

The second data set corresponds to the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes; see Proschan (1963). Table 4 provides descriptive statistics for the Boeing data set. From this table, note the right skewed nature and high kurtosis level of the data distribution.

Table 4. Summary statistics for the Boeing data.

n	Min.	Median	Mean	Max.	SD	CV	CS	CK
188	1	54	92.074	603	107.916	117.205%	2.122	4.938

Table 5 reports the ML estimates, computed by the EM algorithm and SEs for the BS, EBS, EtBS and ECNBS model parameters. In addition, we report the log-likelihood

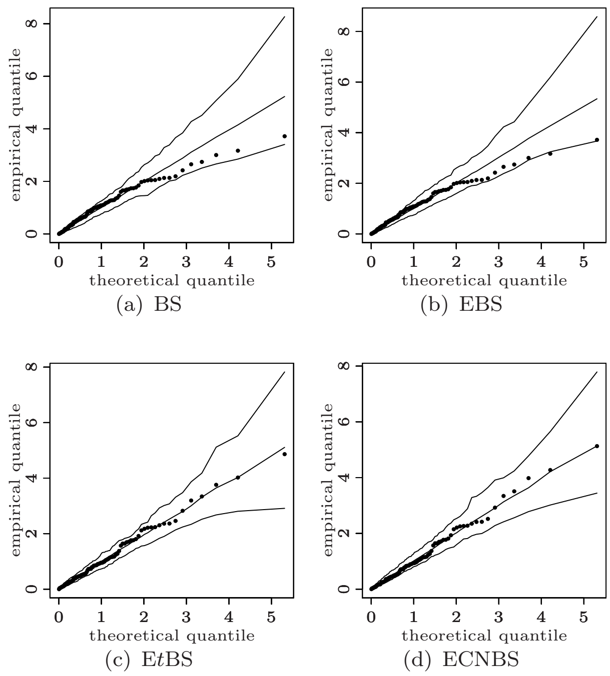


Figure 4. QQ plot and its envelope for the GCS in the indicated distribution with the psi21 data.

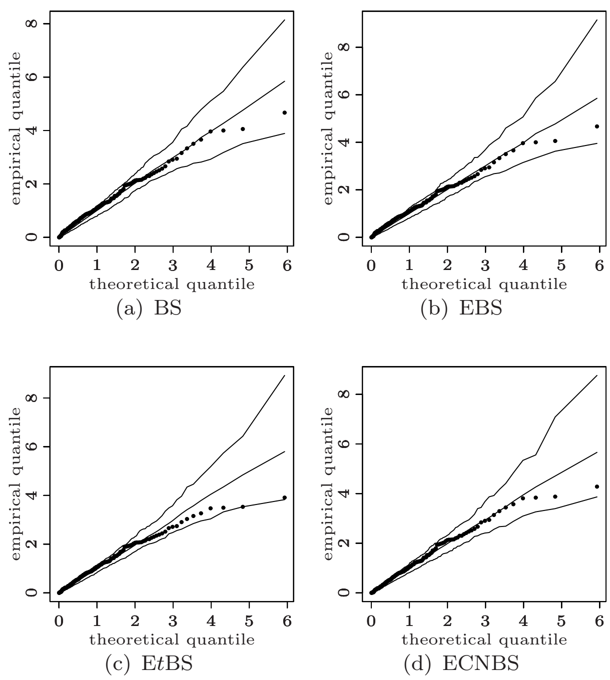


Figure 5. QQ plot and its envelope for the GCS in the indicated distribution with the Boeing data.

and LR values. From Table 5, note that the EtBS and ECNBS models provide better adjustments compared to the other models based on the values of log-likelihood and LR. The QQ plots with simulated envelope of the GCS residuals for the models considered in Table 5 are shown in Figure 5. This figure shows the good agreement of the GCS residuals with the EXP(1) distribution.

Table 5. ML estimates (with SE in parentheses) determined from the EM algorithm for the indicated parameters with the Boeing data.

	Existing models		Proposed models	
Parameter	BS	EBS	EtBS	ECNBS
α	1.5147	1.5147	1.3254	1.2988
	(1.2397)	(1.2397)	(1.0081)	(1.0605)
β	41.3240	41.3240	45.2541	46.9091
	(33.8373)	(33.8398)	(37.0704)	(38.4346)
ε	–	0.0030	0.0010	0.0010
	–	(0.0009)	(0.0008)	(0.0008)
ν	–	–	7.0000	0.0400
γ	–	–	–	0.1100
$\ell(\widehat{\boldsymbol{\theta}})$	−1041.845	−1041.843	−1036.124	−1034.526
LR	–	0.004	11.496	14.638

5. Concluding remarks

In this work, we have introduced an asymmetric generalization of the scale-mixture Birnbaum-Saunders distribution. The new family of distributions is based on the epsilon-skew-normal distribution and provides more flexibility in terms of skewness and kurtosis (heavy tails). Moreover, it allows the computation of the maximum likelihood estimates of the model parameters by using the EM algorithm. A Monte Carlo simulation study was carried out to evaluate the behaviour of the maximum likelihood estimators of the corresponding parameters. We have applied the proposed scale-mixture epsilon Birnbaum-Saunders models to two real-world data sets. We have also derived analytically the observed information matrix, which facilitates the computation of the standard errors of the estimates. As part of future work, it would be of interest to extend the proposed models to the multivariate case. Moreover, regression models based on the proposed distributions with right-censored survival data can be considered; see Lachos et al. (2017). Work on these problems is currently under progress and we hope to report these findings in a future paper.

Acknowledgement

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Appendix. Observed information matrix

Let $\boldsymbol{I}_{\mathrm{SEBS}}(\boldsymbol{\theta})$ be the observed information matrix of the SEBS$(\alpha,\beta,\varepsilon,H)$ model, where $\boldsymbol{\theta} = (\alpha,\beta,\varepsilon)^\top$. Now, let $T = t$ be an observation from a $T \sim$ SEBS$(\alpha,\beta,\varepsilon,H)$ RV with log-likelihood

$$\ell(\boldsymbol{\theta}) \propto -\log(\alpha)-\frac{1}{2}\log(\beta)+\log(t+\beta)+\log\left(f_{\mathrm{SMN}}\left(\frac{a(t)}{1+\varepsilon}\right)\right)\mathrm{I}_{\{t<\beta\}}+\log\left(f_{\mathrm{SMN}}\left(\frac{a(t)}{1-\varepsilon}\right)\right)\mathrm{I}_{\{t\geq\beta\}}.$$

Then, $\boldsymbol{I}_{\mathrm{SEBS}}(\boldsymbol{\theta})$ is given by

$$\boldsymbol{I}_{\mathrm{SEBS}}(\boldsymbol{\theta}) = \begin{pmatrix} I_{\theta_1\theta_1} & I_{\theta_1\theta_2} & I_{\theta_1\theta_3} \\ & I_{\theta_2\theta_2} & I_{\theta_2\theta_3} \\ & & I_{\theta_3\theta_3} \end{pmatrix},$$

where $I_{\theta_i\theta_j} = -\partial^2\ell(\boldsymbol{\theta})/\partial\theta_i\theta_j$, $i,j = 1,2,3$, with $\theta_1 = \alpha$, $\theta_2 = \beta$ and $\theta_3 = \varepsilon$. Next, we provide the second derivatives for each distribution discussed in Section 2.2.

EtBS model

Let $T = t$ be an observation from an EtBS$(\alpha,\beta,\nu,\varepsilon)$ distribution. Then, its log-likelihood function is

$$\begin{aligned}\ell(\boldsymbol{\theta},t) = &-\frac{3}{2}\log(t) + \log(t+\beta) - \log(2) - \log(\alpha) - \frac{1}{2}\log(\beta)\\ &+ \log\{\Gamma(\nu+1)/2\} - \frac{1}{2}\log(\nu) - \frac{1}{2}\log(\pi) - \log\{\Gamma(\nu/2)\}\\ &- \frac{\nu+1}{2}\log\left(1 + \frac{1}{\nu\alpha^2(1+\varepsilon)^2}\left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right)\mathrm{I}_{\{t<\beta\}}\\ &- \frac{\nu+1}{2}\log\left(1 + \frac{1}{\nu\alpha^2(1-\varepsilon)^2}\left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right)\mathrm{I}_{\{t\geq\beta\}},\end{aligned}$$

where $\boldsymbol{\theta} = (\alpha,\beta,\nu,\varepsilon)$, so that its second partial derivatives are given by

$$\begin{aligned}\frac{\partial^2\ell(\boldsymbol{\theta},t)}{\partial\alpha^2} =& \frac{1}{\alpha^2} - \left(\frac{\nu+1}{2}\right)\left\{\frac{2(t-\beta)^2(3\alpha^2\nu\beta t + 6\alpha^2\nu\beta\varepsilon t + 3\alpha^2\varepsilon^2\nu\beta t + (t-\beta)^2)}{\alpha^2(\alpha^2\nu\beta t + 2\alpha^2\nu\beta\varepsilon t + \alpha^2\varepsilon^2\nu\beta t + (t-\beta)^2)}\right\}\mathrm{I}_{\{t<\beta\}}\\ &- \left(\frac{\nu+1}{2}\right)\left\{\frac{2(t-\beta)^2(3\alpha^2\nu\beta t - 6\alpha^2\nu\beta\varepsilon t + 3\alpha^2\varepsilon^2\nu\beta t + (t-\beta)^2)}{\alpha^2(\alpha^2\nu\beta t - 2\alpha^2\nu\beta\varepsilon t + \alpha^2\varepsilon^2\nu\beta t + (t-\beta)^2)}\right\}\mathrm{I}_{\{t\geq\beta\}},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\ell(\boldsymbol{\theta},t)}{\partial\alpha\partial\beta} =& \left(\frac{\nu+1}{2}\right)\left\{\frac{2\alpha\nu t(\beta^2-t^2)(\varepsilon+1)^2}{(\alpha^2\nu\beta t + 2\alpha^2\nu\beta\varepsilon t + \alpha^2\varepsilon^2\nu\beta t + (t-\beta)^2)^2}\right\}\mathrm{I}_{\{t<\beta\}}\\ &+ \left(\frac{\nu+1}{2}\right)\left\{\frac{2\alpha\nu t(\beta^2-t^2)(\varepsilon-1)^2}{(\alpha^2\nu\beta t - 2\alpha^2\nu\beta\varepsilon t + \alpha^2\varepsilon^2\nu\beta t + (t-\beta)^2)^2}\right\}\mathrm{I}_{\{t\geq\beta\}},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\ell(\boldsymbol{\theta},t)}{\partial\alpha\partial\varepsilon} =& -\left(\frac{\nu+1}{2}\right)\left\{\frac{4\alpha\nu\beta t(\varepsilon+1)(t-\beta)^2}{(\alpha^2\nu\beta t + 2\alpha^2\nu\beta\varepsilon t + \alpha^2\varepsilon^2\nu\beta t + (t-\beta)^2)^2}\right\}\mathrm{I}_{\{t<\beta\}}\\ &- \left(\frac{\nu+1}{2}\right)\left\{\frac{4\alpha\nu\beta t(\varepsilon-1)(t-\beta)^2}{(\alpha^2\nu\beta t - 2\alpha^2\nu\beta\varepsilon t + \alpha^2\varepsilon^2\nu\beta t + (t-\beta)^2)^2}\right\}\mathrm{I}_{\{t\geq\beta\}},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\ell(\boldsymbol{\theta},t)}{\partial\beta^2} =& -\left(\frac{\nu+1}{2}\right)\left\{\frac{2t^3\alpha^2\nu\beta + 4t^3\alpha^2\nu\beta\varepsilon + 2t^3\alpha^2\varepsilon^2\nu\beta + t^4 + 4t^2\beta^2 - 4t^3\beta - \beta^4}{\beta^2(\alpha^2\nu\beta t + 2\alpha^2\nu\beta\varepsilon t + \alpha^2\varepsilon^2\nu\beta t + (t-\beta)^2)^2}\right\}\mathrm{I}_{\{t<\beta\}}\\ &- \left(\frac{\nu+1}{2}\right)\left\{\frac{2t^3\alpha^2\nu\beta - 4t^3\alpha^2\nu\beta\varepsilon + 2t^3\alpha^2\varepsilon^2\nu\beta + t^4 + 4t^2\beta^2 - 4t^3\beta - \beta^4}{\beta^2(\alpha^2\nu\beta t - 2\alpha^2\nu\beta\varepsilon t + \alpha^2\varepsilon^2\nu\beta t + (t-\beta)^2)^2}\right\}\mathrm{I}_{\{t\geq\beta\}}\\ &- \frac{1}{(t+\beta)^2} + \frac{1}{2\beta^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\ell(\boldsymbol{\theta},t)}{\partial\beta\partial\varepsilon} =& \left(\frac{\nu+1}{2}\right)\left\{\frac{2\alpha^2\nu t(\varepsilon+1)(\beta^2-t^2)}{(\alpha^2\nu\beta t + 2\alpha^2\nu\beta\varepsilon t + \alpha^2\varepsilon^2\nu\beta t + (t-\beta)^2)^2}\right\}\mathrm{I}_{\{t<\beta\}}\\ &+ \left(\frac{\nu+1}{2}\right)\left\{\frac{2\alpha^2\nu t(\varepsilon-1)(\beta^2-t^2)}{(\alpha^2\nu\beta t - 2\alpha^2\nu\beta\varepsilon t + \alpha^2\varepsilon^2\nu\beta t + (t-\beta)^2)^2}\right\}\mathrm{I}_{\{t\geq\beta\}},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\ell(\boldsymbol{\theta},t)}{\partial\varepsilon^2} =& -\left(\frac{\nu+1}{2}\right)\left\{\frac{2(t^2-2\beta t+\beta^2)(3\alpha^2\nu\beta t + 6\alpha^2\nu\beta\varepsilon t + 3\alpha^2\varepsilon^2\nu\beta t + (t-\beta)^2)}{(\alpha^2\nu\beta t + 2\alpha^2\nu\beta\varepsilon t + \alpha^2\varepsilon^2\nu\beta t + (t-\beta)^2)^2}\right\}\mathrm{I}_{\{t<\beta\}}\\ &- \left(\frac{\nu+1}{2}\right)\left\{\frac{2(t^2-2\beta t+\beta^2)(3\alpha^2\nu\beta t - 6\alpha^2\nu\beta\varepsilon t + 3\alpha^2\varepsilon^2\nu\beta t + (t-\beta)^2)}{(\alpha^2\nu\beta t - 2\alpha^2\nu\beta\varepsilon t + \alpha^2\varepsilon^2\nu\beta t + (t-\beta)^2)^2}\right\}\mathrm{I}_{\{t\geq\beta\}}.\end{aligned}$$

ECNBS MODEL

Let $T = t$ be an observation from an ECNBS$(\alpha,\beta,\varepsilon,\gamma)$ distribution. Then, its log-likelihood function is

$$\begin{aligned}\ell(\boldsymbol{\theta},t) &= -\frac{3}{2}\log(t)+\log(t+\beta)-\log(2)-\log(\alpha)-\frac{1}{2}\log(\beta)\\ &+\log\left(\nu\phi\left(\frac{a(t)}{(1+\varepsilon)};0,\frac{1}{\gamma}\right)+(1-\nu)\phi\left(\frac{a(t)}{(1+\varepsilon)};0,1\right)\right)\mathrm{I}_{\{t<\beta\}}\\ &+\log\left(\nu\phi\left(\frac{a(t)}{(1-\varepsilon)};0,\frac{1}{\gamma}\right)+(1-\nu)\phi\left(\frac{a(t)}{(1-\varepsilon)};0,1\right)\right)\mathrm{I}_{\{t\geq\beta\}},\end{aligned}$$

where $\boldsymbol{\theta}=(\alpha,\beta,\varepsilon,\nu,\gamma)$, $\phi\left(\frac{a(t)}{(1+\varepsilon)};0,\frac{1}{\gamma}\right)=\phi(\alpha,\beta,\varepsilon,\gamma)$ and $\phi\left(\frac{a(t)}{(1-\varepsilon)};0,1\right)=\phi(\alpha,\beta,\varepsilon)$. The second partial derivatives are given by

$$\begin{aligned}\frac{\partial^2\ell(\boldsymbol{\theta},t)}{\partial\alpha^2} =& -\frac{\left(\nu\phi^{(1,0,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(1,0,0)}(\alpha,\beta,\varepsilon)\right)^2}{(\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon))^2}\mathrm{I}_{\{t\geq\beta\}}\\ &+\frac{\nu\phi^{(2,0,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(2,0,0)}(\alpha,\beta,\varepsilon)}{\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon)}\mathrm{I}_{\{t\geq\beta\}}\\ &+\frac{\nu\phi^{(2,0,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(2,0,0)}(\alpha,\beta,\varepsilon)}{\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon)}\mathrm{I}_{\{t<\beta\}}\\ &-\frac{\left(\nu\phi^{(1,0,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(1,0,0)}(\alpha,\beta,\varepsilon)\right)^2}{(\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon))^2}\mathrm{I}_{\{t<\beta\}}+\frac{1}{\alpha^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\ell(\boldsymbol{\theta},t)}{\partial\alpha\partial\beta} =& -\frac{\left(\nu\phi^{(0,1,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(0,1,0)}(\alpha,\beta,\varepsilon)\right)\left(\nu\phi^{(1,0,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(1,0,0)}(\alpha,\beta,\varepsilon)\right)}{(\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon))^2}\mathrm{I}_{\{t\geq\beta\}}\\ &+\frac{\nu\phi^{(1,1,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(1,1,0)}(\alpha,\beta,\varepsilon)}{\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon)}\mathrm{I}_{\{t\geq\beta\}}\\ &+\frac{\nu\phi^{(1,1,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(1,1,0)}(\alpha,\beta,\varepsilon)}{\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon)}\mathrm{I}_{\{t<\beta\}}\\ &-\frac{\left(\nu\phi^{(0,1,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(0,1,0)}(\alpha,\beta,\varepsilon)\right)\left(\nu\phi^{(1,0,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(1,0,0)}(\alpha,\beta,\varepsilon)\right)}{(\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon))^2}\mathrm{I}_{\{t<\beta\}},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\ell(\boldsymbol{\theta},t)}{\partial\alpha\partial\varepsilon} =& \frac{\left(\nu\phi^{(0,0,1,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(0,0,1)}(\alpha,\beta,\varepsilon)\right)\left(\nu\phi^{(1,0,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(1,0,0)}(\alpha,\beta,\varepsilon)\right)}{(\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon))^2}\mathrm{I}_{\{t\geq\beta\}}\\ &+\frac{-\nu\phi^{(1,0,1,0)}(\alpha,\beta,\varepsilon,\gamma)-(1-\nu)\phi^{(1,0,1)}(\alpha,\beta,\varepsilon)}{\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon)}\mathrm{I}_{\{t\geq\beta\}}\\ &+\frac{\nu\phi^{(1,0,1,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(1,0,1)}(\alpha,\beta,\varepsilon)}{\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon)}\mathrm{I}_{\{t<\beta\}}\\ &-\frac{\left(\nu\phi^{(0,0,1,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(0,0,1)}(\alpha,\beta,\varepsilon)\right)\left(\nu\phi^{(1,0,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(1,0,0)}(\alpha,\beta,\varepsilon)\right)}{(\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon))^2}\mathrm{I}_{\{t<\beta\}},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\ell(\boldsymbol{\theta},t)}{\partial\beta^2} =& -\frac{\left(\nu\phi^{(0,1,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(0,1,0)}(\alpha,\beta,\varepsilon)\right)^2}{(\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon))^2}\mathrm{I}_{\{t\geq\beta\}}\\ &+\frac{\nu\phi^{(0,2,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(0,2,0)}(\alpha,\beta,\varepsilon)}{\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon)}\mathrm{I}_{\{t\geq\beta\}}\\ &+\frac{\nu\phi^{(0,2,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(0,2,0)}(\alpha,\beta,\varepsilon)}{\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon)}\mathrm{I}_{\{t<\beta\}}\\ &-\frac{\left(\nu\phi^{(0,1,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(0,1,0)}(\alpha,\beta,\varepsilon)\right)^2}{(\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon))^2}\mathrm{I}_{\{t<\beta\}}+\frac{1}{2\beta^2}-\frac{1}{(\beta+t)^2},\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 \ell(\boldsymbol{\theta},t)}{\partial\beta\partial\varepsilon} =& \frac{\left(\nu\phi^{(0,0,1,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(0,0,1)}(\alpha,\beta,\varepsilon)\right)\left(\nu\phi^{(0,1,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(0,1,0)}(\alpha,\beta,\varepsilon)\right)}{(\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon))^2}\mathrm{I}_{\{t\geq\beta\}} \\
&+ \frac{-\nu\phi^{(0,1,1,0)}(\alpha,\beta,\varepsilon,\gamma)-(1-\nu)\phi^{(0,1,1)}(\alpha,\beta,\varepsilon)}{\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon)}\mathrm{I}_{\{t\geq\beta\}} \\
&+ \frac{\nu\phi^{(0,1,1,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(0,1,1)}(\alpha,\beta,\varepsilon)}{\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon)}\mathrm{I}_{\{t<\beta\}} \\
&- \frac{\left(\nu\phi^{(0,0,1,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(0,0,1)}(\alpha,\beta,\varepsilon)\right)\left(\nu\phi^{(0,1,0,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(0,1,0)}(\alpha,\beta,\varepsilon)\right)}{(\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon))^2}\mathrm{I}_{\{t<\beta\}},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \ell(\boldsymbol{\theta},t)}{\partial\varepsilon^2} =& -\frac{\left(-\nu\phi^{(0,0,1,0)}(\alpha,\beta,\varepsilon,\gamma)-(1-\nu)\phi^{(0,0,1)}(\alpha,\beta,\varepsilon)\right)^2}{(\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon))^2}\mathrm{I}_{\{t\geq\beta\}} \\
&+ \frac{\nu\phi^{(0,0,2,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(0,0,2)}(\alpha,\beta,\varepsilon)}{\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon)}\mathrm{I}_{\{t\geq\beta\}} \\
&+ \frac{\nu\phi^{(0,0,2,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(0,0,2)}(\alpha,\beta,\varepsilon)}{\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon)}\mathrm{I}_{\{t<\beta\}} \\
&- \frac{\left(\nu\phi^{(0,0,1,0)}(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi^{(0,0,1)}(\alpha,\beta,\varepsilon)\right)^2}{(\nu\phi(\alpha,\beta,\varepsilon,\gamma)+(1-\nu)\phi(\alpha,\beta,\varepsilon))^2}\mathrm{I}_{\{t<\beta\}}.
\end{aligned}
$$

ESLBS model

Let $T=t$ be an observation from an ESLBS$(\alpha,\beta,\varepsilon,\nu)$ distribution. Then, its likelihood function is

$$
\begin{aligned}
\ell(\boldsymbol{\theta},t) =& \log(\nu)-\frac{3}{2}\log(t)+\log(t+\beta)-\log(2)-\log(\alpha)-\frac{1}{2}\log(\beta) \\
&+ \log\left(\int_0^1 u^{\nu-1}\phi\left(\frac{a(t)}{1+\varepsilon};0,\frac{1}{u}\right)\mathrm{d}u\right)\mathrm{I}_{\{t<\beta\}} \\
&+ \log\left(\int_0^1 u^{\nu-1}\phi\left(\frac{a(t)}{1-\varepsilon};0,\frac{1}{u}\right)\mathrm{d}u\right)\mathrm{I}_{\{t\geq\beta\}}
\end{aligned}
$$

where $\boldsymbol{\theta}=(\alpha,\beta,\varepsilon,\nu,\gamma)$, $\phi\left(\frac{a(t)}{1+\varepsilon};0,\frac{1}{u}\right)=\phi(\alpha,\beta,\varepsilon,u)$ and $\phi\left(\frac{a(t)}{1-\varepsilon};0,\frac{1}{u}\right)=\phi(\alpha,\beta,\varepsilon,u)$. The second partial derivatives are given by

$$
\begin{aligned}
\frac{\partial^2 \ell(\boldsymbol{\theta},t)}{\partial\alpha^2} =& -\frac{\left(\int_0^1 \nu u^{\nu-1}\phi^{(1,0,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}{\left(\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}\mathrm{I}_{\{t\geq\beta\}} + \frac{\int_0^1 \nu u^{\nu-1}\phi^{(2,0,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}\mathrm{I}_{\{t\geq\beta\}} \\
&+ \frac{\int_0^1 \nu u^{\nu-1}\phi^{(2,0,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}\mathrm{I}_{\{t<\beta\}} - \frac{\left(\int_0^1 \nu u^{\nu-1}\phi^{(1,0,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}{\left(\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}\mathrm{I}_{\{t<\beta\}} + \frac{1}{\alpha^2},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \ell(\boldsymbol{\theta},t)}{\partial\alpha\partial\beta} =& -\frac{\left(\int_0^1 \nu u^{\nu-1}\phi^{(0,1,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)\int_0^1 \nu u^{\nu-1}\phi^{(1,0,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\left(\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}\mathrm{I}_{\{t\geq\beta\}} \\
&+ \frac{\int_0^1 \nu u^{\nu-1}\phi^{(1,1,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}\mathrm{I}_{\{t\geq\beta\}} + \frac{\int_0^1 \nu u^{\nu-1}\phi^{(1,1,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}\mathrm{I}_{\{t<\beta\}} \\
&- \frac{\left(\int_0^1 \nu u^{\nu-1}\phi^{(0,1,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)\int_0^1 \nu u^{\nu-1}\phi^{(1,0,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\left(\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}\mathrm{I}_{\{t<\beta\}},
\end{aligned}
$$

$$\frac{\partial^2\ell(\boldsymbol{\theta},t)}{\partial\alpha\partial\varepsilon} = -\frac{\left(\int_0^1 \nu\left(-u^{\nu-1}\right)\phi^{(0,0,1,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)\int_0^1 \nu u^{\nu-1}\phi^{(1,0,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\left(\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}\mathrm{I}_{\{t\geq\beta\}}$$
$$+\frac{\int_0^1 \nu\left(-u^{\nu-1}\right)\phi^{(1,0,1,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}\mathrm{I}_{\{t\geq\beta\}}+\frac{\int_0^1 \nu u^{\nu-1}\phi^{(1,0,1,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}\mathrm{I}_{\{t<\beta\}}$$
$$-\frac{\left(\int_0^1 \nu u^{\nu-1}\phi^{(0,0,1,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)\int_0^1 \nu u^{\nu-1}\phi^{(1,0,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\left(\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}\mathrm{I}_{\{t<\beta\}},$$

$$\frac{\partial^2\ell(\boldsymbol{\theta},t)}{\partial\beta^2} = -\frac{\left(\int_0^1 \nu u^{\nu-1}\phi^{(0,1,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}{\left(\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}\mathrm{I}_{\{t\geq\beta\}}+\frac{\int_0^1 \nu u^{\nu-1}\phi^{(0,2,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}\mathrm{I}_{\{t\geq\beta\}}$$
$$+\frac{\int_0^1 \nu u^{\nu-1}\phi^{(0,2,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}\mathrm{I}_{\{t<\beta\}}-\frac{\left(\int_0^1 \nu u^{\nu-1}\phi^{(0,1,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}{\left(\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}\mathrm{I}_{\{t<\beta\}}$$
$$+\frac{1}{2\beta^2}-\frac{1}{(\beta+t)^2},$$

$$\frac{\partial^2\ell(\boldsymbol{\theta},t)}{\partial\beta\partial\varepsilon} = -\frac{\left(\int_0^1 \nu\left(-u^{\nu-1}\right)\phi^{(0,0,1,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)\int_0^1 \nu u^{\nu-1}\phi^{(0,1,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\left(\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}\mathrm{I}_{\{t\geq\beta\}}$$
$$+\frac{\int_0^1 \nu\left(-u^{\nu-1}\right)\phi^{(0,1,1,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}\mathrm{I}_{\{t\geq\beta\}}+\frac{\int_0^1 \nu u^{\nu-1}\phi^{(0,1,1,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}\mathrm{I}_{\{t<\beta\}}$$
$$-\frac{\left(\int_0^1 \nu u^{\nu-1}\phi^{(0,0,1,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)\int_0^1 \nu u^{\nu-1}\phi^{(0,1,0,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\left(\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}\mathrm{I}_{\{t<\beta\}},$$

$$\frac{\partial^2\ell(\boldsymbol{\theta},t)}{\partial\varepsilon^2} = -\frac{\left(\int_0^1 \nu\left(-u^{\nu-1}\right)\phi^{(0,0,1,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}{\left(\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}\mathrm{I}_{\{t\geq\beta\}}+\frac{\int_0^1 \nu u^{\nu-1}\phi^{(0,0,2,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}\mathrm{I}_{\{t\geq\beta\}}$$
$$+\frac{\int_0^1 \nu u^{\nu-1}\phi^{(0,0,2,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}{\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u}\mathrm{I}_{\{t<\beta\}}-\frac{\left(\int_0^1 \nu u^{\nu-1}\phi^{(0,0,1,0)}(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}{\left(\int_0^1 \nu u^{\nu-1}\phi(\alpha,\beta,\varepsilon,u)\,\mathrm{d}u\right)^2}\mathrm{I}_{\{t<\beta\}}.$$

References

Andrews, D.R., Mallows, C.L. (1974). Scale mixtures of normal distributions. Journal of the Royal Statistical Society, Series B 36, 99-102.

Arellano-Valle, R.B., Gómez, H.W., Quintana, F.A. (2005). Statistical inference for a general class of asymmetric distributions. Journal of Statistical Planning and Inference 128, 427-443.

Arellano-Valle, R.B., Cortés, M.A., Gómez, H.W. (2010). An extension of the epsilon-skew-normal distribution. Communications in Statistics-Theory and Methods 39, 912-922.

Balakrishnan, N., Leiva, V., Sanhueza, A., Vilca, F. (2009). Estimation in the Birnbaum-Saunders distribution based on scale-mixture of normals. Statistics and Operations Research Transactions 33, 171-192.

Balakrishnan, N., Gupta, R.C., Kundu, D., Leiva, V., Sanhueza, A. (2011). On some mixture models based on the Birnbaum-Saunders distribution and associated inference. Journal of Statistical Planning and Inference 141, 2175-2190.

Bhatti, C.R. (2010). The Birnbaum-Saunders autoregressive conditional duration model. Mathematics and Computers in Simulation 80, 2062-2078.

Birnbaum, Z.W., Saunders, S.C. (1969). A new family of life distributions. Journal of Applied Probability 6, 319-327.

Branco, M., Dey, D. (2001). A general class of multivariate skew-elliptical distribution. Journal of Multivariate Analysis 79, 93-113.

Castillo, N.O., Gómez, H.W., Bolfarine, H. (2011). Epsilon Birnbaum-Saunders distribution family: properties and estimation. Statistical Papers 52, 871-883.

Castro, L.M., Gómez, H.W., Valenzuela, M. (2012). Epsilon half-normal model: properties and inference. Computational Statistics & Data Analysis 56, 4338-4347.

Contreras-Reyes, J.E., Arellano-Valle, R.B. (2013). Growth estimates of cardinal fish (Epigonus crassicaudus) based on scale mixtures of skew-normal distributions. Fisheries Research 147, 137-144.

Dempster, A.P., Laird, N.M., Rubin, D.B. (1977). Maximum likelihood from incomplete data via the EM algorithm. Journal of the Royal Statistical Society, Series B 39, 1-38.

Díaz-García, J.A., Leiva, V. (2005). A new family of life distributions based on elliptically contoured distributions. Journal of Statistical Planning and Inference 128, 445-457.

Fang, K.T., Kotz, S., Ng, K.W. (1990). Symmetric Multivariate and Related Distributions. Chapman & Hall, London.

Lachos, V.H., Dey, D.K., Cancho, V.G., Louzada, F. (2017). Scale mixtures log-Birnbaum-Saunders regression models with censored data: A Bayesian approach. Journal of Statistical Computation and Simulation 87, 2002-2022.

Labra, F.V., Garay, A.M., Lachos, V.H., Ortega, E.M.M. (2012). Estimation and diagnostics for heteroscedastic nonlinear regression models based on scale mixtures of skew-normal distributions. Journal of Statistical Planning and Inference 142, 2149-2165.

Leao, J., Leiva, V., Saulo, H., Tomazella, V. (2017). Birnbaum-Saunders frailty regression models: Diagnostics and application to medical data. Biometrical Journal 59, 291-314.

Leiva, V., Marchant, C., Saulo, H., Aslam, M., Rojas, F. (2014). Capability indices for Birnbaum-Saunders processes applied to electronic and food industries. Journal of Applied Statistics 41, 1881-1902.

Mudholkar, G.S., Hutson, A.D. (2000). The epsilon-skew-normal distribution for analyzing near-normal data. Journal of Statistical Planning and Inference 83, 291-309.

Ng, H.K.T., Kundu, D., Balakrishnan, N. (2003). Modified moment estimation for the two-parameter Birnbaum-Saunders distribution. Computational Statistics & Data Analysis 43, 283-298.

Paula, G.A., Leiva, V., Barros, M., Liu, S. (2012). Robust statistical modeling using the Birnbaum-Saunders-t distribution applied to insurance. Appl. Applied Stochastic Models in Business and Industry 28, 16-34.

Proschan, F. (1963). Theoretical explanation of observed decreasing failure rate. Technometrics 15, 375-383.

Rieck, J.R., Nedelman, J.R. (1991). A log-linear model for the Birnbaum-Saunders distribution. Technometrics 33, 51-60.

Vilca, F., Leiva, V. (2006). A new fatigue life model based on the family of skew-elliptical distributions. Communications in Statistics-Theory and Methods 35, 229-244.

Zeller, C.B., Lachos, V.H., Vilca-Labra, F.E. (2011). Local influence analysis for regression models with scale mixtures of skew-normal distributions. Journal of Applied Statistics 38, 343-368.