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Letter to the Editor

Letter to the Editor on the paper

"Stochastic representations and a geometric parametrization of the two-dimensional Gaussian law"

by Dietrich, Kalke, and Richter, published in the Chilean Journal of Statistics, Vol. 4, No. 2, September 2013, 27-59 [comment on MR3120428].

by Christian Rau*

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Abstract

With regard to the above named paper we clarify an issue regarding coupling, and make some further remarks on correlation, and cone measure.

Keywords: Cone measure · Correlation · Couplings.

Mathematics Subject Classification: Primary 60D05 · Secondary 62H20.

1. Introduction

In February 2014, the captioned paper was sent to me by the American Mathematical Society (AMS) for the purpose of writing a review of it for Mathematical Reviews©. While I agree that the approach to the two-dimensional Gaussian distribution from this paper, in particular the parametrization, is novel and valuable, a point of confusion arose which I deemed important enough to prompt me to write this letter; this issue is pointed out and clarified in Section 2. In Section 3, I point out a connection between the paper and the notion of cone measures. All notation, page numbers, and references refer to the article itself, unless otherwise stated.

2. Correlation Coefficient and Couplings

On page 44, the authors present as formula (11) upper and lower bounds for the standard (Pearson) correlation coefficient ρ of a bivariate normal distribution with mean zero and variances a^2, b^2, where $a, b > 0$ and $a \neq b$. After this, they interject this remark: "For a

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basic study of the closely related notion of a maximal correlation for given marginals we refer to Fréchet (1957) and Höffding (1940)". The authors appear to refer here to what are known in the literature as the Höffding-Fréchet bounds, which are upper and lower bounds for bivariate distribution functions with given marginals; see e.g. (Rachev and Rüschendorf, 1998, Sec. 3.1). In line with common terminology, we concisely refer to these bivariate distributions as *couplings* of the said marginals. The upper Höffding-Fréchet bound yields an upper bound for the covariance of any two variables X, Y for which the covariance is defined (Rachev and Rüschendorf, 1998, formula (3.1.3), p. 108). However, there is no relation between (11) and the latter upper bound: while the discussion containing (11) assumes that the bivariate distribution has a normal density (that is, it is 'regular' in the terminology of Dietrich et al.), one allows in the set of couplings the *deterministic couplings*, where $Y = T(X)$ with measurable T, as defined in (Villani, 2009, Def. 1.2, Ch. 1). To make this point still more explicit, note that the case $X \sim \text{Normal}(0, a^2)$ and $Y = \frac{b}{a}X$ is excluded in Dietrich et al., but allowed in (Rachev and Rüschendorf, 1998, a formula (3.1.3), p. 108), and will give (after some computations for the extreme rightmost expression, which can for example be performed with Mathematica©) equality in the right-hand inequality there; here of course $\rho = 1$, and nothing of interest is obtained by the coupling approach.

We close this section with some comments of a more subjective nature. If $(X, Y)^\top$ is replaced by $(cX, Y)^\top$ with $c > 0$, the correlation ρ between the components does not change, while it does so in the case of the angular descriptor α of Dietrich et al. It seems to us that the goals of how well parameters describe a 'shape' of a distribution is inextricably linked to the notion of what kind of transformations are supposed to leave that 'shape' unchanged. The transformation from two sentences earlier is natural for a statistician, as it amounts only to a change in unit of measurement (and as such preferably does not affect the parameter), but it is less natural for a geometer.

3. Cone Measure

In Section 2.4 (pp. 34–35), the authors do not mention that the distribution of $\mathcal{W}_{a,b}$ is that of normalized cone measure on the ellipse. This follows by comparing the formula for the area of a sector of an ellipse with the formula for $\varphi_{EP}(x, y)$ given below (3) on p. 35 in Dietrich et al. For the definition and recent results about cone measures, see Naor (2007).

References

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Reply to the letter of Christian Rau

By Wolf-Dieter Richter

1. It was proved in Hoeffding (1940) that the correlation coefficient of a bivariate cdf H attains lower and upper bounds if H equals certain functions H^- and H^+, respectively. Fréchet (1951) proved that, correspondingly, H^- and H^+ are lower and upper bounds for H which are called now Fréchet-Hoeffding bounds of a bivariate cdf. In statistical analysis, e.g., the maximal Hoeffding correlation coefficient is often of interest.

The mentioned general correlation inequalities in mind, our formula (11) gives new insight into lower and upper bounds for the correlation which we derived with a view toward the specific Gaussian case. Our discussion following this formula is aimed to always being aware of the classical general results when possibly dealing with other specific cases. Such cases could concern, e.g., elliptically contoured, $l_{n,p}$-symmetric or star-shaped distributions. Couplings as suggested by C.Rau, where the one component of a random vector under consideration is a deterministic function of the other one, were actually not in the scope of our paper.

2. If $(X, Y)^T$ is replaced by $(cX, Y)^T$ with $c > 0$ then both the angular descriptor α and the scaling descriptor b^2/a^2 of the shape of the distribution's density level sets are jointly transformed. Because the correlation coefficient is invariant w.r.t. such joint transformations it is indeed also of interest from a geometric point of view.

3. The consideration of our paper is essentially based upon a non-Euclidean geometric measure representation and a stochastic representation of a corresponding random vector in Richter (2011). Making use of a non-Euclidean arc-length measure allows, e.g., to avoid elliptical integrals in solving certain problems in elliptically contoured distribution theory and in dealing with certain measurements of ellipses. The nothing but trivial role which non-Euclidean geometry plays in this research area is not mentioned in the review MR3120428, possibly a circumstance of subjective nature. In his letter to the editor the author does not explain any additional insight for the reader of our paper coming out from the additional interpretation of our arc-length measure in terms of area contents as the cone measure does. In our opinion,

such additional interpretation is not necessary for explaining the two main topics of the paper. Moreover, the connection between our arc-length measure and a sector (or cone) measure is described in Remark 4.3 in Richter (2011).

By the way, I had several times and for different reasons the opportunity to refer to sector and cone measures and some work from this research area in Richter (2009) and the additional

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