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CONTENTS

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Statistical Process Control
Research Paper

Control chart for monitoring the mean in symmetric data

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Abstract

The control charts are the main tools used for monitoring quality characteristic. Usually the monitored characteristic is the process mean and the most used control charts for such monitoring are Shewhart $\overline{X}$, CUSUM and EWMA, which are based on two assumptions: independence between monitored samples and that the monitored variable follows a normal distribution. However, deviations from any of these assumptions imply poor control chart performance. Considering this, the present work proposes a control chart to monitoring the mean, based on the bootstrap method, for data that follows a distribution that belongs to the symmetric class of distributions. Simulation studies are performed for the proposed method, in order to evaluate the in-control and the out-of-control average run length, to evaluate the behavior of the control limits and to compare the proposed method with the traditional Shewhart $\overline{X}$. The simulation study indicates that the proposed approach presents better in-control average run length than the usual Shewhart $\overline{X}$. Regarding the power of detection, the proposed method presents good performance, being comparable to Shewhart $\overline{X}$, but with the advantage of a better in-control average run length. Practical use of the proposed approach is illustrated with a real example of pH of red wines.

Keywords: Bootstrap · Heavy-tailed distribution · Light-tailed distribution · Statistical Process control · Symmetric distributions.

Mathematics Subject Classification: 62P30 · 62F99

1. Introduction

Competition in manufacturing industries has been growing around the world to achieve ever higher quality standards. Naveed et al. (2020) mentioned that the main concern of the companies is to maintain a positive reputation in the market. The authors also state that a key aspect to enable this goal is through the quality of the products. In this context,

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statistical process control (SPC) is a powerful set of techniques to meet this end. More specifically, control charts are the most common tools in SPC used to monitor processes. The control chart proposed by Shewhart (1931), called the $\overline{X}$ chart, is the most known and used SPC technique. The standard assumptions of this technique are: (i) the collections of independent samples over time, (ii) the monitored control characteristic follows a normal distribution. This method has the purpose of detecting shifts in the mean of magnitude greater than 1.5 standard deviation of the mean.

Chakraborti et al. (2008) emphasized the need to ascertain precisely if the monitored data follow the assumptions of the employed chart. However, in many practical situations, when verifying whether the data follow the standard assumptions, it is considered that the data are normally distributed just because they have a symmetrical shape. Schilling and Nelson (1976), Borror et al. (1999), Calzada and Scariano (2001), and Noorossana et al. (2011) commented that when the monitored data do not follow a normal distribution, the usual Shewhart $\overline{X}$ chart shows low performance in the monitoring, by generating more false alarms or by not detecting deviations from the true mean with the usual precision (da Silva et al., 2019).

It is in this context of low performance of the usual charts, under non-normal symmetrical distribution (Rezac et al., 2015), that arise alternatives methods to monitoring symmetrical data. For example, correction factors in quantiles of the distribution or in the form of control limits of the usual method (Bai and Choi, 1995; Tadikamalla and Popescu, 2007; Tadikamalla et al., 2008). Other approaches considered to solve the problem of non-normality are the non-parametric techniques (Haq and Khoo, 2019; Willemain and Runger, 1996; Chakraborti et al., 2001)), and data transformation (Qiu and Zhang, 2015). There are also alternative procedures based on quantiles of distributions with a heavier tail than the tail of the normal distribution (Calzada and Scariano, 2001; Tsai et al., 2005; Zhang et al., 2009) and control charts via intensive computational methods (Bajgier, 1992; Seppala et al., 1995; Liu and Tang, 1996).

More recently, Ahmed et al. (2019) proposed a technique based on a more comprehensive class of distribution, known as the long-tailed symmetric (LTS), but in the context of small and moderate mean deviations. Nonetheless, there is a wider class of distributions, in which the LTS is a particular case, the symmetrical class of distributions or univariate elliptical (Berkane and Bentler, 1986; Fang et al., 1990; Rao, 1990).

Considering this scenario, this work has the objective of proposing, via parametric bootstrap method, a monitoring chart for the process mean for changes greater than 1.5 standard deviations. The underling feature of the data distribution is the symmetric one. Besides, we focus on a wide class of symmetrical distribution known as the symmetrical distribution class. The in-control and out-of-control average run length (ARL_0 and ARL_1, respectively) of the proposed method are evaluated through simulations and compared with the ARL_0 and ARL_1 of the standard Shewhart $\overline{X}$ chart in different scenarios of the symmetrical class. The practical application of the method is illustrated by monitoring the pH of red wines. Finally, we argue that having a proposed method that provides a general framework for any member of the symmetric distribution class, regardless of the tail weight, leads to a better decision making.

This paper is organized as follows: this introductory section. Section 2 presents briefly the symmetrical class of distributions. Section 3 presents the proposed approach for monitoring changes in the process mean in symmetric data. Section 4 presents the Average Run Length (ARL) performance of the proposed charts under different combinations of the model parameters based on simulation studies. Application to a real data set is presented in Section 5. Final considerations are reported in Section 6.

2. Symmetrical distribution class

In this section, we presents background on the symmetrical class of distributions.

Let X be a random variable, with $X \in \mathbb{R}$. The distribution of X belongs to the class of symmetric distributions with location parameter $\mu \in \mathbb{R}$ and scale parameter $\phi > 0$, if its probability density function is of the form:

$$f(x;\mu,\phi) = \frac{1}{\sqrt{\phi}} h\left(\frac{(x-\mu)^2}{\phi}\right), \quad x \in \mathbb{R},$$

for some function $h(u) > 0$, for $u > 0$, such that $\int_0^\infty u^{-\frac{1}{2}} h(u)\mathrm{d}u = 1$. The conditions imposed on h, guarantee that $f(x;\mu,\phi)$ is, in fact, a probability density function. The function h is called the density-generating function and it may depend on other parameters than μ and ϕ, which is the case of the Student-t and power-exponential distributions, for example.

We denote $X \sim \mathrm{S}(\mu,\phi)$, if X belongs to the symmetric distributions class of parameters μ and ϕ. Some examples of distributions that belong to this class are shown in Table 1, as presented in Medeiros and Ferrari (2017), there are distributions with heavier tails (for example, Student-t distributions and type II logistic) and lighter tails (for example, power-exponential distribution with $-1 < \kappa < 0$ and type I logistic) than the normal distribution. Moreover, the class of symmetric distributions considers also bimodal distributions such as the generalized Kotz distribution.

Let us assume that $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \xi\phi$ exist, for some constant $\xi > 0$. Furthermore, if $X \sim \mathrm{S}(\mu,\phi)$, then $a + bX \sim \mathrm{S}(a + b\mu, b^2\phi)$ with $a \in \mathbb{R}$ and $b \in \mathbb{R} - \{0\}$, that is, the distribution of any linear transformation of a random variable, which its distribution belongs to the symmetrical class, also belongs to the symmetrical class. Particularly, if $Z = (X-\mu)/\sqrt{\phi}$, then $Z \sim \mathrm{S}(0,1)$ and the probability density function of Z is given by

$$f(z) = f(z;0,1) = h(z^2), \quad z \in \mathbb{R},$$

where h is the density generating function of X. In order to estimate the parameters of these models, we adopt the maximum likelihood method. For more details on properties, demonstrations and theoretical results for the symmetric distribution class (Berkane and Bentler, 1986; Fang et al., 1990; Rao, 1990).

3. Proposed approach

In this section, we propose a control chart using a parametric bootstrap method for a class of symmetric distributions. Our method is based on the work of Bajgier (1992) and Liu and Tang (1996), who used a non-parametric approach. Here, we make some modifications to use on a parametric bootstrap, since we are establishing theoretical results. When the distribution of the data is correctly identified, we generate samples of the suitable distribution in order to capturing the real nature of the data. As commented in Efron and Tibshirani (1994) and Davison and Hinkley (1997), when we fit a suitable distribution, the parametric bootstrap provides better results to estimate the quantiles, in our case the control limits, than the non-parametric bootstrap.

Before establishing results for the proposed control charts, we need to specify some notation and quantities such as sample size (m), subsample size (n), the frequency (s) (assumed here as $s = 1$), statistic used in the monitoring (in this case, $\overline{X}$), and lower and upper control limits (LCL and UCL). When $\overline{X} >$ UCL or $\overline{X} <$ LCL, an action to search

Table 1. Density-generating function and ξ values, for some symmetric distributions.

Distribution	$h(u), \quad u > 0$	ξ
Normal	$\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-u/2}$	1
Student-t	$\frac{\nu^{\nu/2}}{B(1/2,\nu/2)}(\nu+u)^{-\frac{\nu+1}{2}},\ \nu > 0$	$\frac{\nu}{\nu-2},\ \nu > 2$
Type I logistic	$c\frac{\mathrm{e}^{-u}}{(1+\mathrm{e}^{-u})^2},\ c \approx 1.4843$	0.7957
Type II logistic	$\frac{\mathrm{e}^{-\sqrt{u}}}{\left(1+\mathrm{e}^{-\sqrt{u}}\right)^2}$	$\frac{\pi^2}{3}$
Kotz	$\frac{r^{(2N-1)/2}}{\Gamma\left(\frac{2N-1}{2}\right)},\ r > 0, N \geq 1$	$\frac{2N-1}{2r}$
Power-exponential [1]	$\frac{1}{c(\kappa)}\mathrm{e}\left\{-\frac{1}{2}u^{1/(1+\kappa)}\right\}, -1 < \kappa \leq 1$	$2^{1+\kappa}\frac{\Gamma(1{,}5(1+\kappa))}{\Gamma(\frac{1+\kappa}{2})}$

where Γ is the gamma functions and $c(\kappa) = \Gamma(1+\frac{1+\kappa}{2})2^{1+(1+\kappa)/2}$.

for special causes in the process must be taken. Thus, when the process is in-control, it is desirable to have few false alarms to reduce the number of unnecessary stops in the process. In SPC, the usual metric to measure the performance of a control chart is the ARL until an out-of-control point is detected. When the process is in control, a large ARL_0 is desirable. Let us choose α, the probability of a type I error such that

$$\alpha = P(\overline{X} > \mathrm{UCL} \mid \mu = \mu_0) + P(\overline{X} < \mathrm{LCL} \mid \mu = \mu_0) \tag{1}$$

with μ_0 the value of μ when the process is in-control.

On other hand, if the process is out-of-control, it is desirable that the control chart signals very soon, that is, a low ARL_1. The power of a control chart expressed as $1-\beta$ is

$$P(\overline{X} > \mathrm{UCL} \mid \mu = \mu_1) + P(\overline{X} < \mathrm{LCL} \mid \mu = \mu_1), \tag{2}$$

where $\mu_1 = \mu_0 + \delta\,\sigma/\sqrt{m}$ is the mean when the process is out-of-control, σ is the standard deviation of the characteristic of interest, δ is the shift size expressed in units of the standard deviation of the mean and β is the probability of a type II error. Moreover,

$$\mathrm{ARL}_0 = \frac{1}{\alpha}$$

and

$$\mathrm{ARL}_1 = \frac{1}{1-\beta}.$$

Note that from Equations (1) and (2) the control limits can be seen as quantiles of the distribution of the statistic used to monitor the process that provide a certain probability (α or $1-\beta$). Thus, by fixing either (α or $1-\beta$) or (ARL_0 or ARL_1), we get the control limits as the quantiles of the mean distribution. That said, based on the bootstrap method for control charts, proposed by Bajgier (1992) and Gandy and Kvaløy (2013), we obtain the control limits to monitor the process mean according to Algorithm 1.

Algorithm 1 Control limits to monitor the process mean.

1: Generate a $n \times m$ observation matrix of the considered symmetrical distribution, where n is the size of the subsample. Calculate

$$\overline{X}_i = \frac{1}{n} \sum_{l=(i-1)n+1}^{in} X_l, \quad i = 1, \dots, m.$$

2: Using the empirical distribution of $\overline{X}$, obtained using the samples of the symmetrical distribution in Step 1, obtain the quantiles of order $\alpha/2$ and $1-\alpha/2$, referred here, respectively, as $\widehat{q}_{\frac{\alpha}{2}}$ and $\widehat{q}_{(1-\frac{\alpha}{2})}$.

3: Repeat Step 1 and 2 B times using the quantities

$$\widehat{\mathrm{LCL}} = \frac{1}{B}\sum_{i=1}^{B} \widehat{q}_{\frac{\alpha}{2},i} \qquad \text{and} \qquad \widehat{\mathrm{UCL}} = \frac{1}{B}\sum_{i=1}^{B} \widehat{q}_{(1-\frac{\alpha}{2}),i}$$

as the control limits.

Algorithm 1, described previously, can be schematically depicted as

Replicate 1	Replicate 2	...	Replicate B		
$\overline{x}_1$	$\overline{x}_1$	...	$\overline{x}_1$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$		
$\overline{x}_m$	$\overline{x}_m$	...	$\overline{x}_m$		
$\widehat{q}_{\frac{\alpha}{2},1}$	$\widehat{q}_{\frac{\alpha}{2},2}$	...	$\widehat{q}_{\frac{\alpha}{2},B}$	$\rightarrow$	$\widehat{\mathrm{LCL}} = \frac{1}{B}\sum_{i=1}^{B} \widehat{q}_{\frac{\alpha}{2},i}$
$\widehat{q}_{(1-\frac{\alpha}{2}),1}$	$\widehat{q}_{(1-\frac{\alpha}{2}),2}$	...	$\widehat{q}_{(1-\frac{\alpha}{2}),B}$	$\rightarrow$	$\widehat{\mathrm{UCL}} = \frac{1}{B}\sum_{i=1}^{B} \widehat{q}_{(1-\frac{\alpha}{2}),i}$.

Since this is a computationally intensive method, in order to obtain the control limits, we recommend simulated samples of sizes $m \geq 2,000$ and $B \geq 5,000$, aiming to obtain more accurate results, without possible bootstrap quantile bias. These values of m and B are suggested based on previous studies; see for more on the suggested values of m and B in Davison and Hinkley (1997). In practical situations, where the parameters of the process distribution are most likely unknown, we recommend the following procedure to estimate the control limits:

(1) Obtain observed values $\boldsymbol{x} = (x_1, \dots, x_k)^\top$ (from sample of size k, under statistical control, of your population of interest) and adjust possible models for the data. Use the AIC (Akaike information criterion) and BIC (Bayesian information criterion) for selected model and goodness-of-fit techniques in order to evaluate the chosen distribution;

(2) If the most suitable model belongs to the symmetric class, use $\widehat{\boldsymbol{\theta}}(\boldsymbol{x})$, the maximum likelihood estimate of the parameters of the distribution, identified in Step (1), to obtain the control limits using Algorithm 1.

For more details on the computationally intensive and resampling procedure, see Davison and Hinkley (1997).

4. Chart performance evaluation and comparison

In this section, a detailed simulation study is conducted in order to gain insight into the detection abilities in the proposed control charts when we compare with the usual Shewhart charts.

The simulation study considers two distributions, Student-t and power-exponential. The Student-t distribution is usually used as an alternative to the normal distribution when the behavior of the data suggests a symmetrical distribution, but with tails heavier than the normal distribution. Lange et al. (1989) commented that the Student-t model can be seen as a robust parametric extension of the normal model, since it allows to reduce the influence of aberrant observations. On top of that, the Student-t allows the adjustment of the kurtosis of the data distribution through the ν parameter, which represents its degrees of freedom. For the purpose of evaluating the performance of the proposed chart, values of $\nu = \{3, 5, 10\}$ and 20 are considered in the simulation study. Additionally, the power-exponential distribution is also used because its κ parameter allows it to have both lighter and heavier tails than the normal distribution, thus making it a good alternative for non-normal symmetric data. The simulation study has lighter tail ($\kappa = \{-0.45, -0.25\}$) and heavier tail ($\kappa = \{0.3, 0.4\}$) scenarios than the normal distribution.

Furthermore, the scenarios are evaluated considering the following subsample size: $n = \{1, 2, 3, 10, 100, 500\}$. Bearing in mind that the proposed method is compared to the usual Shewhart chart, and we consider deviations in the mean of $\delta = \{0.0, \mp 1.5, \mp 2.0, \mp 3.0\}$ standard deviations. The analyzes presented below are based on the assessments of the ARL_0, ARL_1 and the asymptotic behavior of the control limits, represented by $n = 100$ and 500. The parameter settings μ and ϕ adopted are intended to simulate several practical situations such as small (1, 2, 5) and large (100 and 200) values of process mean, in addition to considering the data dispersion index, which is given by

$$I_d = \frac{\text{Var}(X)}{\text{E}(X)}$$

and represents the variability of the data in relation to the mean. Regarding the dispersion index, we considered four categories: very underdispersed ($I_d \approx 0.033$), moderate underdispersed ($I_d \approx 0.67$), moderate overdispersed ($I_d \approx 1.67$) and very overdispersed ($I_d \approx 3.33$). The target value set for ARL_0 is 370.40 samples, which is equivalent to $\alpha = 0.0027$ (reference value of the Shewhart $\overline{X}$ chart).

The computational routines were developed using the `R` software (R Core Team, 2018) version 3.6.2 for Windows platform and are available at:

https://github.com/lucasdofs/Control-Chart-Symmetrical.

The Shewhart LCL and UCL considered in the simulation are given by

$$\text{LCL} = \mu_0 - \frac{\sigma_0}{\sqrt{n}} \quad \text{and} \quad \text{UCL} = \mu_0 + \frac{\sigma_0}{\sqrt{n}},$$

where μ_0 and σ_0 are the in-control mean and in-control standard deviation, respectively.

Algorithm 2 proposes a way for estimating ARL_0 and ARL_1.

Algorithm 2 Procedure to estimate ARL_0 and ARL_1.

1: Generate $\boldsymbol{x}_{(n\times 5000)} = (x_1, \ldots, x_{n\times 5000})^\top$, a column vector of size $n \times 5000$ of the distribution of interest, and calculate

$$\overline{X}_i = \frac{1}{n}\sum_{l=(i-1)n+1}^{in} X_l, \quad i = 1, \ldots, 5000.$$

2: The control limits are compared with the 5000 sample-shifted mean (shifted factor $= \delta \times \sigma_0/\sqrt{n}$ and store the position of the first out of control sample, in which the value of the sample-shifted mean is higher than UCL or lower than LCL.

3: Steps 1 and 2 are repeated 10000 times independently, and ARL_0 or ARL_1 is calculated based on the average of the positions obtained in Step 2.

The diagram below illustrates Algorithm 2:

Replicate 1	Replicate 2	$\ldots$	Replicate 10000
$\overline{x}_1 + \delta\frac{\sigma_0}{\sqrt{n}}$	$\overline{x}_1 + \delta\frac{\sigma_0}{\sqrt{n}}$	$\ldots$	$\overline{x}_1 + \delta\frac{\sigma_0}{\sqrt{n}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\overline{x}_{5000} + \delta\frac{\sigma_0}{\sqrt{n}}$	$\overline{x}_{5000} + \delta\frac{\sigma_0}{\sqrt{n}}$	$\ldots$	$\overline{x}_{5000} + \delta\frac{\sigma_0}{\sqrt{n}}$
a_1	a_2	$\ldots$	a_{10000}

Then, we estimate ARL_0 and ARL_1 by means of

$$\widehat{\text{ARL}}_j = \frac{\sum_{i=1}^{10000} a_i}{10000}, \quad j = 0, 1,$$

where a_i, for $i = 1, \ldots, 10000$, represents the position of the first sample in which the value of the sample mean, plus δ standard deviations, is higher than UCL or lower than LCL.

Tables 2 to 5 and Tables 6 to 9 present the results of the computational study carried out for the Student-t and power-exponential distributions, respectively. The estimated results are expressed in terms of the quantities $\widehat{\text{ARL}}_0$, $\widehat{\text{ARL}}_1$ and the control limits, in addition, in parentheses are $\widehat{\text{ARL}}_0$, $\widehat{\text{ARL}}_1$ for the usual Shewhart control limits.

Table 2. Control limits, $\widehat{\text{ARL}}_0$ and $\widehat{\text{ARL}}_1$ of the proposed method, considering the Student-t with $\nu = 3$ (in parentheses are the $\widehat{\text{ARL}}_0$ and $\widehat{\text{ARL}}_1$ for the usual Shewhart limits).

				δ						
Parameters	n	$\widehat{\text{LCL}}$	$\widehat{\text{UCL}}$	-3.0	-2.0	-1.5	0	1.5	2.0	3.0
$\mu = 100.00.\ \phi = 1.00$ $\sigma^2 = 3.00.\ I_d = 0.03$	1	90.91	109.07	64.95 (4.08)	164.1 (10.72)	228.45 (23.58)	361.92 (72.68)	231.61 (23.52)	163.78 (10.75)	64.99 (4.08)
	2	94.05	105.95	4.89 (1.56)	49.31 (2.48)	114.55 (7.78)	349.38 (79.85)	110.93 (7.72)	48.71 (2.45)	4.87 (1.07)
	3	95.36	104.63	1.31 (1.01)	12.18 (1.48)	49.36 (3.37)	356.23 (85.81)	49.54 (3.30)	11.92 (1.35)	1.31 (1.01)
	10	97.78	102.22	1.00 (1.00)	1.02 (1.00)	1.28 (1.00)	360.63 (112.37)	1.28 (1.00)	1.01 (1.00)	1.00 (1.00)
	100	99.41	100.58	1.00 (1.00)	1.00 (1.00)	1.03 (1.00)	375.41 (175.90)	1.03 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	99.75	100.25	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	372.30 (273.51)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 9.00.\ \phi = 2.00$ $\sigma^2 = 6.00.\ I_d = 0.67$	1	-3.82	21.81	64.37 (2.25)	162.81 (10.67)	226.73 (23.37)	351.00 (75.69)	220.81 (24.18)	158.93 (11.05)	62.39 (2.12)
	2	0.58	17.40	4.98 (1.06)	49.48 (2.47)	113.92 (7.77)	348.40 (79.97)	112.77 (7.76)	48.99 (2.46)	4.83 (1.07)
	3	2.44	15.58	1.32 (1.00)	12.03 (1.48)	49.55 (4.43)	360.44 (85.84)	50.98 (4.43)	12.58 (1.50)	1.32 (1.00)
	10	5.85	12.15	1.00 (1.00)	1.01 (1.00)	1.29 (1.00)	351.47 (110.05)	1.28 (1.00)	1.01 (1.00)	1.00 (1.00)
	100	8.18	9.82	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	358.47 (188.19)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	8.66	9.34	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	352.07 (228.52)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 2.00.\ \phi = 1.00$ $\sigma^2 = 3.00.\ I_d = 1.50$	1	-7.08	11.04	65.10 (2.02)	161.00 (11.04)	224.12 (23.95)	349.92 (72.56)	226.8 (23.83)	160.31 (10.86)	63.45 (1.98)
	2	-3.94	7.95	4.88 (1.04)	49.01 (2.46)	113.9 (7.76)	349.15 (79.81)	113.53 (7.53)	48.74 (2.47)	4.92 (1.00)
	3	-2.64	6.64	1.31 (1.01)	12.18 (1.37)	49.65 (3.32)	354.25 (84.79)	49.65 (3.31)	12.18 (1.37)	1.31 (1.00)
	10	-0.23	4.23	1.00 (1.00)	1.01 (1.00)	1.27 (1.00)	358.13 (110.50)	1.30 (1.00)	1.01 (1.00)	1.00 (1.00)
	100	1.42	2.58	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	361.87 (178.28)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	1.76	2.24	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	365.46 (271.31)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 3.00.\ \phi = 3.00$ $\sigma^2 = 9.00.\ I_d = 3.00$	1	-12.72	18.70	64.45 (2.01)	162.61 (10.57)	226.07 (24.34)	354.34 (71.58)	224.67 (23.21)	161.73 (10.74)	63.65 (1.98)
	2	-7.28	13.29	4.88 (1.00)	49.05(3.21)	113.15(8.12)	354.46 (79.59)	112.42 (7.98)	48.27 (2.59)	4.93 (1.08)
	3	-5.04	11.04	1.32 (1.00)	12.23 (1.24)	49.34 (4.02)	358.24 (84.78)	49.85 (3.31)	12.54 (1.12)	1.32 (1.02)
	10	-0.86	6.87	1.00 (1.00)	1.01 (1.00)	1.30 (1.00)	352.81 (108.88)	1.30 (1.00)	1.02 (1.00)	1.00 (1.00)
	100	2.00	4.00	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	368.31 (191.45)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	2.66	3.44	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	372.63 (262.68)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)

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Table 3. Control limits, $\widehat{\mathrm{ARL}}_0$ and $\widehat{\mathrm{ARL}}_1$ of the proposed method, considering the Student-t with $\nu = 5$ (in parentheses are the $\widehat{\mathrm{ARL}}_0$ and $\widehat{\mathrm{ARL}}_1$ for the usual Shewhart limits).

				δ						
Parameters	n	$\widehat{\mathrm{LCL}}$	$\widehat{\mathrm{UCL}}$	−3.0	−2.0	−1.5	0	1.5	2.0	3.0
$\mu = 100.00$. $\phi = 2.00$ $\sigma^2 = 3.33$. $I_d = 0.03$	1	92.31	107.69	11.22 (2.03)	55.72 (7.13)	112.49 (17.93)	352.56 (87.86)	112.53 (17.71)	54.76 (7.87)	10.98 (1.97)
	2	95.07	104.93	1.48 (1.00)	7.15 (1.08)	23.83 (5.97)	352.57 (108.37)	23.90 (6.01)	7.13 (2.10)	1.47 (1.00)
	3	96.17	103.83	1.06 (1.01)	2.38 (1.44)	7.42 (3.23)	348.02 (125.75)	7.37 (3.44)	2.32 (1.92)	1.05 (1.00)
	10	98.12	101.88	1.07 (1.00)	1.00 (1.00)	1.00 (1.00)	349.26 (199.15)	1.07 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	99.45	100.55	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	342.54 (342.54)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	99.75	100.25	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	364.02 (364.02)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 5.00$. $\phi = 2.00$ $\sigma^2 = 3.33$. $I_d = 0.67$	1	−2.67	12.68	11.05(2.00)	54.23 (7.98)	108.10 (17.77)	339.16 (85.71)	108.00 (17.83)	54.50 (7.93)	10.98 (2.02)
	2	0.07	9.93	1.46 (1.02)	7.10 (2.35)	23.98 (5.99)	344.71 (126.00)	23.82 (6.02)	7.18 (2.49)	1.44 (1.01)
	3	1.69	8.83	1.06 (1.00)	2.31 (1.00)	7.49 (3.18)	352.36 (125.04)	7.39 (3.00)	2.33 (1.00)	1.06 (1.00)
	10	3.12	6.88	1.07 (1.00)	1.00 (1.00)	1.00 (1.00)	342.91 (193.58)	1.06 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	4.45	5.55	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	341.06 (341.44)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	4.75	5.25	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	363.70 (363.70)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 2.00$. $\phi = 2.00$ $\sigma^2 = 3.33$. $I_d = 1.67$	1	−5.69	9.68	11.41 (2.03)	55.49 (7.95)	110.58 (17.58)	350.81 (84.80)	109.42 (17.05)	55.32 (7.76)	11.29 (1.99)
	2	−2.93	6.93	1.47 (1.01)	7.18 (2.37)	24.03 (5.98)	342.45 (108.11)	24.07 (5.96)	7.17 (2.34)	1.45 (1.02)
	3	−1.83	5.83	1.07 (1.00)	2.37 (1.44)	7.43 (2.99)	353.08 (125.57)	7.51 (3.04)	2.32 (1.43)	1.06 (1.00)
	10	0.12	3.88	1.07 (1.00)	1.00 (1.00)	1.00 (1.00)	344.14 (195.11)	1.07 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	1.45	2.55	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	342.59 (342.65)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	1.75	2.25	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	364.85 (364.85)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 1.00$. $\phi = 2.00$ $\sigma^2 = 3.33$. $I_d = 3.33$	1	−6.68	8.69	11.21 (2.00)	54.34 (7.98)	111.25 (17.82)	354.89 (86.91)	112.53 (17.86)	55.20 (7.99)	10.98 (2.00)
	2	−3.93	5.93	1.21 (1.10)	3.82 (2.35)	11.45 (5.94)	342.45 (108.01)	23.73 (6.05)	7.07 (2.40)	1.47 (1.11)
	3	−2.83	4.83	1.06 (1.00)	2.37 (1.45)	7.52 (3.12)	353.08 (125.33)	7.53 (2.87)	2.42 (2.00)	1.06 (1.00)
	10	−0.87	2.87	1.07 (1.00)	1.00 (1.00)	1.00 (1.00)	350.42 (201.52)	1.07 (1.01)	1.00 (1.00)	1.00 (1.00)
	100	0.45	1.55	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	346.90 (346.90)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	0.75	1.25	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	363.02 (363.02)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)

Table 4. Control limits, $\widehat{\mathrm{ARL}}_0$ and $\widehat{\mathrm{ARL}}_1$ of the proposed method, considering the Student-t with $\nu = 10$ (in parentheses are the $\widehat{\mathrm{ARL}}_0$ and $\widehat{\mathrm{ARL}}_1$ for the usual Shewhart limits).

				δ						
Parameters	n	$\widehat{\mathrm{LCL}}$	$\widehat{\mathrm{UCL}}$	−3.0	−2.0	−1.5	0	1.5	2.0	3.0
$\mu = 40.00$. $\phi = 1.00$ $\sigma^2 = 1.25$. $I_d = 0.03$	1	36.08	43.91	3.88 (2.03)	16.22 (6.93)	41.07 (16.02)	349.80 (137.94)	40.03 (17.02)	15.69 (6.83)	3.38 (2.00)
	2	37.41	42.59	1.98 (1.00)	3.13 (2.33)	8.44 (5.56)	344.73 (185.14)	8.51 (5.60)	3.09 (2.33)	1.20 (1.00)
	3	37.94	42.06	1.02 (1.00)	1.63 (1.12)	3.65 (2.94)	348.76 (211.11)	3.73 (2.93)	1.65 (1.21)	1.02 (1.00)
	10	38.92	41.08	1.00 (1.00)	1.00 (1.00)	1.05 (1.00)	368.43 (309.80)	1.05 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	39.66	40.33	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	360.57 (360.57)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	39.95	40.15	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	368.60 (368.60)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 2.00$. $\phi = 1.00$ $\sigma^2 = 1.25$. $I_d = 0.63$	1	−1.91	5.92	3.39 (2.01)	16.02 (5.84)	40.76 (16.05)	350.40 (137.34)	40.56 (16.05)	16.05 (5.85)	3.41 (1.98)
	2	−0.60	4.59	1.19 (1.02)	3.14 (2.36)	8.67 (5.57)	353.52 (183.05)	8.68 (5.53)	3.13 (2.32)	1.19 (1.00)
	3	−0.06	4.06	1.02 (1.00)	1.66 (1.43)	3.73 (2.94)	347.79 (209.95)	3.74 (2.97)	1.64 (1.49)	1.02 (1.00)
	10	0.92	3.08	1.00 (1.00)	1.00 (1.00)	1.05 (1.00)	362.89 (308.87)	1.05 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	1.66	2.33	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	359.04 (359.04)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	1.85	2.15	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	372.82 (372.82)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 3.00$. $\phi = 4.00$ $\sigma^2 = 5.00$. $I_d = 1.67$	1	−4.68	10.80	3.38 (2.08)	15.69 (6.88)	40.23 (17.23)	350.53 (138.37)	40.46 (16.98)	16.15 (6.22)	3.42 (2.42)
	2	−2.19	8.19	1.19 (1.02)	3.16 (1.98)	8.64 (5.45)	353.80 (183.80)	8.58 (5.39)	3.12 (2.43)	1.19 (1.03)
	3	−1.12	7.12	1.02 (1.00)	1.62 (1.00)	3.69 (2.81)	349.12 (215.11)	3.69 (2.90)	1.63 (1.00)	1.02 (1.00)
	10	0.84	5.16	1.00 (1.00)	1.00 (1.00)	1.05 (1.00)	348.58 (302.91)	1.05 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	2.33	3.67	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	379.51 (379.51)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	2.70	3.30	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	362.18 (362.18)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 2.00$. $\phi = 5.00$ $\sigma^2 = 6.25$. $I_d = 3.13$	1	−6.76	10.76	3.46 (2.04)	16.28 (6.11)	41.22 (17.07)	351.44 (135.79)	40.95 (16.97)	16.02 (5.94)	3.49 (1.99)
	2	−3.80	7.81	1.19 (1.00)	3.22 (1.98)	8.60 (5.21)	351.04 (185.96)	8.60 (4.99)	3.17 (1.49)	1.18 (1.00)
	3	−2.61	6.60	1.03 (1.00)	1.65 (1.02)	3.71 (2.38)	340.71 (211.32)	3.65 (2.61)	1.63 (1.02)	1.03 (1.01)
	10	−0.41	4.41	1.00 (1.00)	1.00 (1.00)	1.05 (1.00)	354.67 (303.27)	1.05 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	1.25	2.75	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	379.97 (379.98)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	1.67	2.33	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	345.63 (345.63)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)

For both distributions considered, as n increases, the control limits get closer to the true value of the mean (process under control), regardless of the ν parameter, for the Student-t distribution, and the κ parameter for the power-exponential distribution. Furthermore, as expected, for the Student-t distribution, when n and ν increase, the sample mean distribution approaches a normal distribution. Thus, the control limits by the proposed method tend to approach the Shewhart's usual control limits. This behavior is also seen for the power-exponential distribution when n increases and κ approaches 0. This fact is noticeable when we observe the proximity of the ARL_0 values and considering the control limits using the proposed method and the usual Shewhart method. Moreover, the observed behavior of the proposed control limits occurs independently of the process dispersion index, thus showing the robustness in relation to this index.

Based on the ARL_0 for heavy-tailed data distribution, see Tables 2 to 5 and Tables 8 and 9, regardless of the scenario, the proposed method presents ARL_0 around 340 to 380

Table 5. Control limits, $\widehat{\mathrm{ARL}}_0$ and $\widehat{\mathrm{ARL}}_1$ of the proposed method, considering the Student-t with $\nu = 20$ (in parentheses are the $\widehat{\mathrm{ARL}}_0$ and $\widehat{\mathrm{ARL}}_1$ for the usual Shewhart limits).

				δ						
Parameters	n	$\widehat{\mathrm{LCL}}$	$\widehat{\mathrm{UCL}}$	-3.0	-2.0	-1.5	0	1.5	2.0	3.0
$\mu = 100.00.\ \phi = 3.00$ $\sigma^2 = 3.33.\ I_d = 0.03$	1	94.12	105.88	2.47 (2.00)	9.35 (6.49)	23.63 (15.54)	370.34 (205.13)	23.33 (15.48)	9.33 (6.54)	2.43 (2.00)
	2	95.99	104.01	1.15 (1.12)	2.59 (2.30)	6.31 (5.30)	338.44 (255.64)	6.22 (5.39)	2.59 (2.34)	1.14 (1.12)
	3	96.77	103.23	1.00 (1.00)	1.10 (1.02)	1.48 (1.13)	350.19 (293.38)	3.13 (1.10)	1.15 (1.01)	1.01 (1.00)
	10	98.27	101.73	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	340.78 (336.88)	1.04 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	99.45	100.55	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	363.70 (363.70)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	99.75	100.25	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	368.64 (368.64)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 5.00.\ \phi = 3.00$ $\sigma^2 = 3.33.\ I_d = 0.67$	1	-0.87	10.87	2.43 (2.02)	9.21 (6.56)	23.22 (15.63)	345.35 (205.13)	23.60 (15.49)	9.33 (6.60)	2.43 (2.00)
	2	0.99	10.10	1.15 (1.12)	2.60 (3.25)	6.32 (5.37)	350.44 (260.74)	6.21 (5.44)	2.55 (2.30)	1.15 (1.12)
	3	3.27	7.73	1.02 (1.02)	1.48 (1.09)	3.04 (2.95)	352.36 (284.71)	3.39 (2.93)	2.33 (1.46)	1.06 (1.02)
	10	3.27	7.73	1.05 (1.00)	1.00 (1.00)	1.00 (1.00)	341.78 (338.52)	1.03 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	4.45	5.55	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	363.48 (368.48)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	4.75	5.25	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	370.81 (370.81)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 2.00.\ \phi = 3.00$ $\sigma^2 = 3.33.\ I_d = 1.67$	1	-3.87	7.87	2.43 (1.99)	9.30 (6.59)	23.22 (15.41)	349.33 (206.78)	23.35 (15.47)	9.19 (6.61)	2.42 (2.03)
	2	-2.00	6.00	1.14 (1.08)	2.55 (2.36)	6.29 (5.41)	342.47 (257.90)	6.25 (5.40)	2.56 (2.33)	1.14 (1.02)
	3	-1.23	5.23	1.02 (1.01)	1.52 (1.46)	3.17 (2.93)	345.04 (285.07)	3.15 (2.95)	1.52 (1.47)	1.01 (1.00)
	10	0.27	3.73	1.03 (1.00)	1.00 (1.00)	1.00 (1.00)	347.54 (343.02)	1.04 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	1.45	2.55	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	347.78 (347.65)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	1.75	2.25	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	370.84 (370.84)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 1.00.\ \phi = 3.00$ $\sigma^2 = 3.33.\ I_d = 3.33$	1	-4.87	6.87	2.43 (2.00)	9.29 (6.74))	23.19 (15.76)	344.41 (205.14)	23.23 (15.17)	9.25 (6.58)	2.49 (2.00)
	2	-3.00	5.00	1.14 (1.00)	2.58 (2.34)	6.31 (5.32)	342.45 (259.21)	6.33 (5.44)	2.56 (2.38)	1.14 (1.01)
	3	-2.23	4.23	1.02 (1.00)	1.52 (1.34)	3.18 (2.83)	345.04 (285.07)	3.15 (2.90)	1.52 (1.39)	1.01 (1.00)
	10	-0.73	2.73	1.05 (1.00)	1.00 (1.00)	1.00 (1.00)	347.54 (343.02)	1.04 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	0.45	1.55	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	371.68 (371.68)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	0.75	1.25	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	370.84 (370.84)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)

Table 6. Control limits, $\widehat{\mathrm{ARL}}_0$ and $\widehat{\mathrm{ARL}}_1$ of the proposed method, considering the power-exponential with $\kappa = -0.45$ (in parentheses are the $\widehat{\mathrm{ARL}}_0$ and $\widehat{\mathrm{ARL}}_1$ for the usual Shewhart limits).

				δ						
Parameters	n	$\widehat{\mathrm{LCL}}$	$\widehat{\mathrm{UCL}}$	-3.0	-2.0	-1.5	0	1.5	2.0	3.0
$\mu = 50.00.\ \phi = 3.00$ $\sigma^2 = 1.52.\ I_d = 0.03$	1	47.03	52.97	1.43 (1.97)	2.69 (5.42)	4.72 (14.95)	331.28 (2453.04)	4.83 (14.94)	2.75 (5.53)	1.47 (2.09)
	2	47.68	52.32	1.06 (1.13)	1.79 (2.31)	3.32 (4.98)	334.62 (1674.89)	3.30 (5.05)	1.76 (2.29)	1.06 (1.12)
	3	48.02	51.98	1.00 (1.01)	1.34 (1.48)	2.35 (2.93)	347.29 (978.20)	2.30 (2.87)	1.34 (1.49)	1.00 (1.01)
	10	48.86	51.14	1.00 (1.00)	1.00 (1.00)	1.03 (1.04)	357.29 (475.48)	1.04 (1.04)	1.00 (1.00)	1.00 (1.00)
	100	49.63	50.37	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	362.80 (429.02)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	49.83	50.17	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	371.77 (371.77)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 3.00.\ \phi = 4.00$ $\sigma^2 = 2.03.\ I_d = 0.67$	1	-0.43	6.43	1.43 (1.98)	2.70 (5.44)	4.74 (15.15)	338.91 (2353.73)	4.76 (14.82)	2.69 (5.50)	1.45 (2.01)
	2	0.32	5.68	1.06 (1.13)	1.77 (2.30)	3.29 (4.96)	342.17 (1629.00)	3.47 (5.09)	1.77 (2.30)	1.06 (1.13)
	3	0.72	5.28	1.01 (1.01)	1.34 (1.48)	2.33 (2.84)	342.95 (975.43)	2.27 (2.86)	1.33 (1.50)	1.01 (1.01)
	10	1.68	4.32	1.00 (1.00)	1.00 (1.00)	1.03 (1.05)	366.09 (463.89)	1.04 (1.04)	1.00 (1.00)	1.00 (1.00)
	100	2.58	3.43	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	360.78 (360.78)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	2.81	3.19	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	372.59 (372.59)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 1.00.\ \phi = 3.00$ $\sigma^2 = 1.52.\ I_d = 1.52$	1	-1.97	3.97	1.43 (2.00)	2.74 (5.42)	4.71 (14.83)	348.63 (2510.84)	4.84 (14.76)	2.79 (5.55)	1.45 (2.04)
	2	-1.32	3.32	1.07 (1.13)	1.80 (2.30)	3.29 (4.97)	353.29 (961.80)	3.28 (5.03)	1.77 (2.27)	1.06 (1.12)
	3	-0.98	2.89	1.00 (1.01)	1.36 (1.50)	2.32 (2.89)	353.29 (961.80)	2.27 (2.86)	1.34 (1.48)	1.00 (1.01)
	10	-0.14	2.14	1.00 (1.00)	1.00 (1.00)	1.04 (1.03)	358.79 (474.93)	1.03 (1.05)	1.00 (1.00)	1.00 (1.00)
	100	0.46	1.54	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	425.74 (425.74)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	0.75	1.25	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	365.57 (365.57)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 1.00.\ \phi = 6.00$ $\sigma^2 = 3.04.\ I_d = 3.04$	1	-3.20	5.20	1.45 (2.01)	2.75 (5.53)	4.89 (14.85)	357.02 (2453.02)	4.79 (14.76)	2.71 (5.48)	1.46 (2.00)
	2	-2.28	4.28	1.07 (1.14)	1.78 (2.31)	3.30 (5.02)	338.52 (1653.62)	3.81 (5.05)	1.79 (2.32)	1.06 (1.13)
	3	-1.79	3.79	1.00 (1.00)	1.33 (1.33)	2.34 (2.33)	342.48 (990.77)	2.31 (2.88)	1.32 (1.48)	1.00 (1.00)
	10	-0.61	2.61	1.00 (1.00)	1.00 (1.00)	1.04 (1.05)	347.92 (450.07)	1.05 (1.04)	1.00 (1.00)	1.00 (1.00)
	100	0.48	1.52	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	393.67 (393.67)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	0.77	1.23	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	365.62 (365.62)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)

samples. Considering subsample sizes $n = \{1, 2, 3\}$ (most used in practical situations) the proposed method presents ARL_0 closer to 370.40 (target value) than the usual Shewhart method (see column $\delta = 0$ of Tables 2 to 5 and Tables 8 and 9). This behavior occurs independently of the dispersion index, showing the flexibility of the method for different situations. Considering the power of detection of the proposed method, for heavy-tailed distributions, some patterns are observed for all scenarios, they are: (i) for a fixed deviation δ, as n increases, the ARL_1 decreases, approaching one sample; (ii) for a fixed subsample size n, as δ increases, ARL_1 decreases, also approaching one sample.

Table 7. Control limits, $\widehat{\mathrm{ARL}}_0$ and $\widehat{\mathrm{ARL}}_1$ of the proposed method, considering the power-exponential with $\kappa = -0.25$ (in parentheses are the $\widehat{\mathrm{ARL}}_0$ and $\widehat{\mathrm{ARL}}_1$ for the usual Shewhart limits).

				δ						
Parameters	n	$\widehat{\mathrm{LCL}}$	$\widehat{\mathrm{UCL}}$	-3.0	-2.0	-1.5	0	1.5	2.0	3.0
$\mu = 20.00.\ \phi = 1.00$ $\sigma^2 = 0.67.\ I_d = 0.03$	1	17.82	22.18	1.63 (2.04)	3.73 (5.83)	7.78 (14.85)	353.72 (1475.83)	7.61 (14.66)	3.68 (5.62)	1.61 (1.98)
	2	18.38	21.62	1.09 (1.13)	1.96 (2.30)	4.01 (5.11)	348.92 (785.06)	3.88 (5.00)	1.96 (2.28)	1.08 (1.12)
	3	18.65	21.35	1.00 (1.00)	1.37 (1.05)	2.56 (2.92)	354.29 (583.45)	2.51 (2.86)	1.37 (1.47)	1.00 (1.00)
	10	19.24	20.76	1.00 (1.00)	1.00 (1.00)	1.03 (1.04)	361.93 (414.45)	1.04 (1.04)	1.00 (1.00)	1.00 (1.00)
	100	19.76	20.24	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	373.80 (373.80)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	19.89	20.11	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	370.77 (370.77)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 1.00.\ \phi = 1.00$ $\sigma^2 = 0.67.\ I_d = 0.67$	1	-1.18	3.18	1.61 (2.01)	3.73 (5.87)	7.74 (14.62)	356.99 (1468.41)	7.75 (14.84)	3.75 (5.85)	1.60 (1.99)
	2	-0.62	2.62	1.08 (1.13)	1.97 (2.30)	3.93 (5.04)	356.25 (783.09)	4.00 (5.11)	1.97 (2.30)	1.09 (1.26)
	3	-0.34	2.35	1.01 (1.01)	1.38 (1.48)	2.52 (2.84)	352.44 (580.43)	2.53 (2.86)	1.39 (1.50)	1.00 (1.01)
	10	0.24	0.76	1.00 (1.00)	1.00 (1.00)	1.03 (1.04)	362.04 (413.75)	1.05 (1.04)	1.00 (1.00)	1.00 (1.00)
	100	0.76	0.24	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	377.50 (377.50)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	0.89	1.11	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	373.80 (373.80)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 2.00.\ \phi = 5.00$ $\sigma^2 = 3.34.\ I_d = 1.67$	1	-2.87	6.87	1.62 (2.01)	3.72 (5.89)	7.72 (14.82)	347.26 (1474.67)	7.62 (14.95)	3.68 (5.90)	1.61 (2.03)
	2	-1.62	5.62	1.08 (1.12)	1.94 (2.30)	3.85 (5.04)	343.57 (765.80)	3.94 (5.11)	1.98 (2.34)	1.08 (1.29)
	3	-1.02	5.02	1.01 (1.01)	1.39 (1.48)	2.52 (2.52)	352.94 (588.16)	2.54 (2.91)	1.39 (1.49)	1.00 (1.00)
	10	0.30	3.70	1.00 (1.00)	1.00 (1.00)	1.04 (1.05)	346.49 (414.75)	1.04 (1.05)	1.00 (1.00)	1.00 (1.00)
	100	0.46	2.54	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	353.19 (353.19)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	1.75	2.25	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	374.40 (374.40)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 1.00.\ \phi = 3.00$ $\sigma^2 = 3.34.\ I_d = 3.34$	1	-3.87	5.87	1.61 (2.00)	3.64 (5.75)	7.55 (14.84)	344.61 (1447.16)	7.61 (14.76)	3.69 (5.80)	1.60 (2.00)
	2	-2.62	4.62	1.08 (1.12)	1.95 (2.34)	3.92 (5.17)	342.42 (774.74)	3.98 (5.08)	1.98 (2.94)	1.09 (1.12)
	3	-2.02	4.02	1.00 (1.00)	1.38 (1.49)	2.50 (2.88)	350.27 (589.64)	2.51 (2.91)	1.38 (1.50)	1.00 (1.00)
	10	-0.70	2.70	1.00 (1.00)	1.00 (1.00)	1.04 (1.05)	353.87 (422.70)	1.03 (1.04)	1.00 (1.00)	1.00 (1.00)
	100	1.46	2.54	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	347.99 (347.99)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	0.75	1.25	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	365.62 (365.62)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)

Table 8. Control limits, $\widehat{\mathrm{ARL}}_0$ and $\widehat{\mathrm{ARL}}_1$ of the proposed method, considering the power-exponential with $\kappa = 0.30$ (in parentheses are the $\widehat{\mathrm{ARL}}_0$ and $\widehat{\mathrm{ARL}}_1$ for the usual Shewhart limits).

				δ						
Parameters	n	$\widehat{\mathrm{LCL}}$	$\widehat{\mathrm{UCL}}$	-3.0	-2.0	-1.5	0	1.5	2.0	3.0
$\mu = 100.00.\ \phi = 2.00$ $\sigma^2 = 3.48.\ I_d = 0.03$	1	93.77	106.23	2.84 (1.99)	11.75 (6.99)	27.49 (15.43)	342.60 (156.80)	27.98 (15.35)	11.64 (6.99)	2.85 (2.00)
	2	95.80	104.20	1.16 (1.00)	2.79 (2.31)	7.24 (5.57)	339.54 (211.36)	7.26 (5.63)	2.82 (2.33)	1.16 (1.00)
	3	96.63	103.37	1.02 (1.00)	1.57 (1.10)	3.37 (1.85)	345.02 (243.33)	2.94 (1.98)	1.46 (1.10)	1.04 (1.00)
	10	98.21	101.78	1.00 (1.00)	1.00 (1.00)	1.05 (1.00)	341.86 (328.73)	1.04 (1.01)	1.00 (1.00)	1.00 (1.00)
	100	99.44	100.56	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	379.83 (379.83)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	99.75	100.25	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	368.01 (368.01)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 5.00.\ \phi = 2.00$ $\sigma^2 = 3.48.\ I_d = 0.69$	1	-1.23	11.23	2.86 (2.00)	11.42 (7.04)	27.82 (15.48)	344.63 (159.19)	27.85 (15.68)	11.73 (7.02)	2.87 (1.98)
	2	0.80	9.20	1.16 (1.00)	2.81 (2.01)	7.13 (4.88)	345.25 (213.12)	7.33 (4.65)	2.84 (1.66)	1.17 (1.00)
	3	1.64	8.36	1.03 (1.00)	1.56 (1.00)	3.41 (2.01)	337.44 (245.43)	3.37 (1.98)	1.56 (1.02)	1.02 (1.00)
	10	3.22	6.78	1.00 (1.00)	1.00 (1.00)	1.05 (1.00)	331.18 (317.76)	1.04 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	4.43	5.57	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	360.24 (360.24)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	4.75	5.25	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	373.15 (373.15)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 1.00.\ \phi = 1.00$ $\sigma^2 = 1.74.\ I_d = 1.74$	1	-3.41	5.41	2.85 (2.00)	11.78 (6.21)	28.40 (15.38)	340.78 (154.31)	28.40 (16.98)	11.79 (6.82)	2.90 (1.99)
	2	-1.97	3.97	1.16 (1.00)	2.76 (2.33)	7.22 (5.49)	343.14 (210.14)	7.16 (4.44)	2.82 (2.01)	1.16 (1.0 0)
	3	-1.38	3.38	1.02 (1.00)	1.56 (1.12)	3.40 (2.88)	345.69 (245.38)	3.31 (2.53)	1.56 (1.24)	1.02 (1.00)
	10	-0.26	3.38	1.00 (1.00)	1.00 (1.00)	1.04 (1.00)	336.31 (317.73)	1.04 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	0.61	1.39	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	365.30 (365.30)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	0.75	1.25	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	370.69 (370.69)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 1.00.\ \phi = 2.00$ $\sigma^2 = 3.48.\ I_d = 3.48$	1	-5.23	7.23	2.84 (2.01)	11.85 (6.33)	28.50 (16.22)	341.77 (156.24)	27.70 (15.99)	11.64 (7.01)	2.86 (2.12)
	2	-3.20	5.20	1.16 (1.00)	2.82 (1.87)	7.23 (3.34)	342.88 (212.48)	7.30 (3.88)	2.81 (1.97)	1.16 (1.00)
	3	-2.36	4.37	1.02 (1.00)	1.57 (1.21)	3.43 (1.87)	346.42 (249.51)	3.43 (2.01)	1.56 (1.21)	1.02 (1.00)
	10	-0.79	2.78	1.00 (1.00)	1.00 (1.00)	1.04 (1.00)	345.99 (319.95)	1.04 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	0.61	1.39	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	361.58 (362.00)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	0.75	1.25	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	363.47 (363.47)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)

For the Student-t distribution in the most extreme scenarios ($\nu = 3$ and 5; see Tables 2 and 3), due to the behavior of the distribution, using the subsample size $n = 1$ does not prove to be the most recommended in these situations. However, when $n = 2$ the ARL_0 reduces considerably (about 50% or more), mainly when $\delta = 3$. In view of this, in situations where the data distribution have very heavy tails, we recommend the use of $n \geq 3$, as these subsample sizes have excellent ARL_0, closer to the target value 370.40 than the usual Shewhart method. Regarding the heavy-tailed power-exponential distribution (Tables 8 and 9), the method shows excellent detection power, compatible with the usual Shewhart chart. Still on ARL_1, for the power-exponential distribution with $0 < \kappa < 1$, it takes, on average, 30 samples to detect a shift of 1.5 standard deviations in the process

Table 9. Control limits, $\widehat{\text{ARL}}_0$ and $\widehat{\text{ARL}}_1$ of the proposed method, considering the power-exponential with $\kappa = 0.40$ (in parentheses are the $\widehat{\text{ARL}}_0$ and $\widehat{\text{ARL}}_1$ for the usual Shewhart limits).

				δ						
Parameters	n	$\widehat{\text{LCL}}$	$\widehat{\text{UCL}}$	-3.0	-2.0	-1.5	0	1.5	2.0	3.0
$\mu = 200.00$. $\phi = 3.00$ $\sigma^2 = 6.38$. $I_d = 0.03$	1	192.27	208.74	3.38 (2.09)	14.31 (8.01)	33.75 (15.95)	343.75 (128.71)	34.42 (16.01)	14.33 (7.87)	3.29 (1.99)
	2	194.19	205.81	1.18 (1.00)	3.06 (2.04)	8.11 (5.98)	339.18 (183.49)	8.24 (5.78)	3.09 (2.09)	1.19 (1.00)
	3	195.37	204.65	1.02 (1.00)	1.61 (1.21)	3.62 (2.44)	350.76 (220.57)	3.58 (2.68)	1.60 (1.10)	1.02 (1.00)
	10	197.57	202.43	1.00 (1.00)	1.00 (1.00)	1.05 (1.00)	341.72 (301.71)	1.04 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	199.25	200.75	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	363.53 (363.53)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	199.66	200.34	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	369.43 (369.43)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 6.00$. $\phi = 2.00$ $\sigma^2 = 4.25$. $I_d = 0.69$	1	-1.13	13.13	3.38 (1.89)	14.49 (7.55)	34.42 (15.92)	352.32 (131.78)	33.83 (15.09)	14.32 (7.82)	3.67 (2.41)
	2	1.25	10.74	1.18 (1.00)	3.03 (1.89)	8.18 (5.87)	344.72 (186.21)	8.28 (5.91)	3.03 (2.01)	1.18 (1.00)
	3	2.22	9.78	1.07 (1.00)	1.63 (1.12)	3.69 (2.56)	344.36 (217.27)	3.61 (2.57)	1.60 (1.10)	1.02 (1.00)
	10	4.02	7.98	1.00 (1.00)	1.00 (1.00)	1.04 (1.00)	338.18 (299.91)	1.04 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	5.38	6.61	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	365.11 (365.11)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	5.72	6.28	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	369.66 (369.66)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 4.00$. $\phi = 3.00$ $\sigma^2 = 6.38$. $I_d = 1.60$	1	-4.73	12.73	3.40 (2.02)	14.33 (7.20)	33.99 (15.50)	344.13 (128.67)	33.75 (16.01)	14.29 (7.92)	3.39 (2.11)
	2	-1.81	9.81	1.17 (1.00)	3.05 (2.15)	8.23 (6.21)	340.18 (184.24)	8.17 (6.64)	3.08 (2.65)	1.18 (1.00)
	3	-0.63	8.63	1.02 (1.00)	1.61 (1.04)	3.61 (2.44)	347.98 (218.82)	3.65 (2.44)	1.62 (1.13)	1.02 (1.00)
	10	1.57	6.43	1.00 (1.00)	1.00 (1.00)	1.04 (1.00)	340.06 (298.09)	1.04 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	3.21	4.76	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	359.28 (359.28)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	3.66	4.33	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	371.28(371.28)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
$\mu = 2.00$. $\phi = 3.00$ $\sigma^2 = 6.38$. $I_d = 3.19$	1	-6.73	10.73	3.31 (2.00)	14.32 (7.11)	34.18 (15.34)	339.57 (126.84)	34.69 (15.42)	14.62 (7.12)	3.36 (2.00)
	2	-3.81	7.81	1.18 (1.00)	3.09 (2.31)	8.30 (5.46)	345.06 (193.59)	8.25 (5.45)	3.09 (2.35)	1.18 (1.00)
	3	-2.63	6.62	1.02 (1.00)	1.61 (1.10)	3.64 (2.65)	336.19 (215.73)	3.68 (2.67)	1.62 (1.00)	1.02 (1.00)
	10	-0.43	4.43	1.00 (1.00)	1.00 (1.00)	1.04 (1.00)	344.24 (304.75)	1.04 (1.00)	1.00 (1.00)	1.00 (1.00)
	100	1.24	2.74	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	360.82 (360.82)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	500	1.66	2.34	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	368.28 (368.28)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)

mean, when using $n = 1$. However, with $n = 2$, the ARL_0 is already reduced to about 8 to 10 samples. This fact only reinforces the excellent applicability of the method for data with heavy tails.

Considering our method for light-tailed distributions (Tables 6 and 7), which furnishes a ARL_0 around 332 to 370 samples, the ARL_0 provided by the usual Shewhart method has higher values (especially when $n < 100$), around 360 to 2400 samples. This peculiar fact of high ARL_0 occurs because the tail of the normal distribution is heavier than that of the power-exponential distribution. Besides, when the process is under control, a sample (from the power-exponential distribution) will rarely exceed the usual Shewhart limits (designed to cover 99.73% of the samples when the process is normally distributed and under control). On the other hand, despite these high ARL_0, the usual procedure shows to detect well large deviations in the process mean, taking a maximum of 16, 6 and 4 samples to detect shifts of 1.5, 2.0 and 3.0 standard deviations, respectively. Moreover, these ARL_0 reduce as n increases. This fact is, at first sight, surprising, as it is expected that the higher the ARL_0, the longer it will take to detect a true alarm. However, as we are in the situation of large shifts (greater than 1.5 standard deviations), these changes in the mean are sufficient for the usual Shewhart chart to indicate such changes satisfactorily. Still for light-tailed distributions, the proposed method meet expectations and provides the ARL_1 within the desired range, since the proposed approach is based on the true distribution of the data (in this case the power-exponential). In addition to that, with regard to the power of detection, the proposed method always presents ARL_1 less than or equal to the usual method. Furthermore, it is worth mentioning that for both types of tail weights, the usual method detects changes in the mean almost instantly ($\text{ARL}_1 = 1$) when $n \geq 10$.

In short, the proposed method presents excellent performance in term of ARL_0 and ARL_1, being the most recommended in cases with distributions of heavier tail than the normal distribution. On the other hand, in the context of light-tailed distributions, the usual Shewhart method is recommended, because although the proposed method has excellent performance, the Shewhart $\overline{X}$ presents ARL_0 equal to or greater than that of the proposed method and it is comparable to the detection power of the proposed approach.

5. A real example

This section illustrates the applicability of the method proposed in Section 3, when the data is symmetric, but not necessarily with normal distribution. The data refers to the result of the pH of 1599 red wines produced by the Portuguese company Vinho Verde, one of the largest wine producers in Portugal, from May 2004 to February 2007. Cortez et al. (2009) provided more details about the Vinho Verde company and the specifications of the data set. The data used are available at `https://archive.ics.uci.edu/ml/datasets/Wine+Quality`. In this section we also consider a comparison of the proposed method to obtain the control limits with the usual Shewhart method.

Based on Cortez et al. (2009), there are strong indications that the 1599 observations come from a process under control. Therefore, the first thousand observations are considered for phase I (process of constructing a control chart). In phase I, we perform a visual graphic analysis (see Figure 1), the descriptive statistics (see Table 10) and a symmetry test using the `symmetry.test` function of the `lawstat` package of `R`, which is based on Miao et al. (2006). In the normal boxplot in Figure 1 we note some possible "atypical" points. However, as the data is under control, it is more likely that these points are just points in the tail of a heavier tailed distribution than the normal distribution. In Table 10, we can observe that the mean are close to the median, differing just around one standard deviation ($\sqrt{0.249} \approx 0.5$). Furthermore, the coefficient of skewness and kurtosis are both close to zero, which is a indication of symmetry. In addition to the strong suggestion of symmetry, observed in Figure 1, and a brief analysis of the descriptive statistics, the symmetry test provides a p-value of 0.74, that is, there is a strong statistical evidence to not reject the hypothesis of data symmetry.

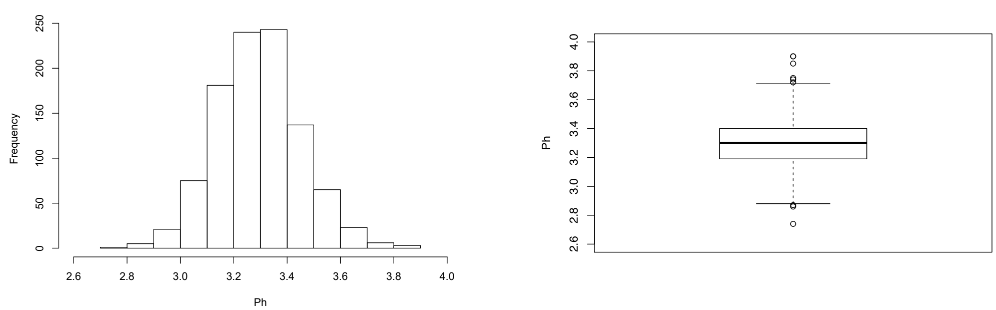

Figure 1. PH Histogram and boxplot of the first thousand wines.

Table 10. Descriptive statistics related to the first thousand pH observations (phase I) of the red wines.

Minimum	Mean	Median	Maximum	Variance	Dispersion index	Skewness	Kurtosis
2.740	3.299	3.300	3.900	0.249	0.008	0.185	0.417

After the assumption of symmetry is considered reasonable, we perform model adequacy tests to find out which symmetric distribution best fits the data. Table 11 shows the models considered, the estimated parameters, AIC and BIC. These measures and estimates are obtained using the `gamlss` package (Stasinopoulos et al., 2007) of the `R` software.

Based on Table 11, we see that the most suitable model for the data is the Student-t model with $\widehat{\mu} = 3.299$, $\widehat{\phi} = 0.007$, $\widehat{\nu} = 2.841$ (which provides an estimated standard error $\widehat{\sigma} = 0.1484$), with the lowest AIC and BIC among the concurrent models. In Figure 2, we present the quantile residuals (Dunn and Smyth, 1996) for the Student-t model, obtained using `gamlss` function. As expected, the quantile residual for the adjusted model are independent and normally distributed, which indicates that the postulated model is reasonable to the data.

Table 11. Parameter estimates, AIC and BIC for the models considered for the pH of red wines.

Distribution	Parameter estimates	AIC	BIC
Normal	$\widehat{\mu} = 3.299,\ \widehat{\phi} = 0.025$	-850.104	-840.288
Student-t	$\widehat{\mu} = 3.299,\ \widehat{\phi} = 0.007,\ \widehat{\nu} = 2.841$	-853.585	-848.862
Power-exponential	$\widehat{\mu} = 3.299,\ \widehat{\phi} = 0.008,\ \widehat{\kappa} = 0.558$	-852.388	-847.665
Type I logistic	$\widehat{\mu} = 3.299,\ \widehat{\phi} = 0.031$	-849.183	-839.367

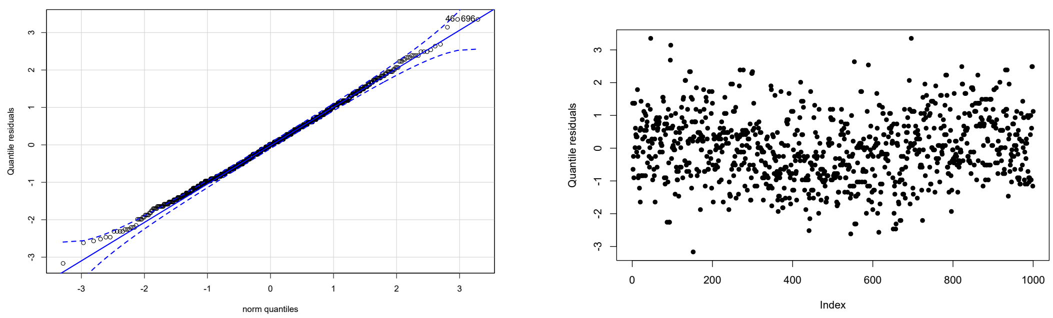

Figure 2. QQ-Plot and dispersion graphic of the quantile residuals for the Student-t model.

Thus, considering the Student-t model and the estimated parameters, we use the procedure described in Section 3 with $n = 1$ (chosen to preserve the original data monitoring scale) and a probability of false alarm equal to 0.0027. For this configuration, we obtain LCL $= 2.50$, UCL $= 4.10$ and an estimated ARL_0 of 373 samples. We see in Figure 3 (left) that the proposed method do not detect any change in the pH of the monitored wines. In contrast, in Figure 3 (right) , we see that, even though the data is under control, the usual method of Shewhart, based on the normal distribution, detects changes in the average pH of the wines, thus generating false alarms. As expected, the proposed method performs better than the usual Shewhart method, in relation to the number of samples until a false alarm, when the data has a heavier tail than the normal distribution (in this case, Student-t distribution with 2.85 degrees of freedom).

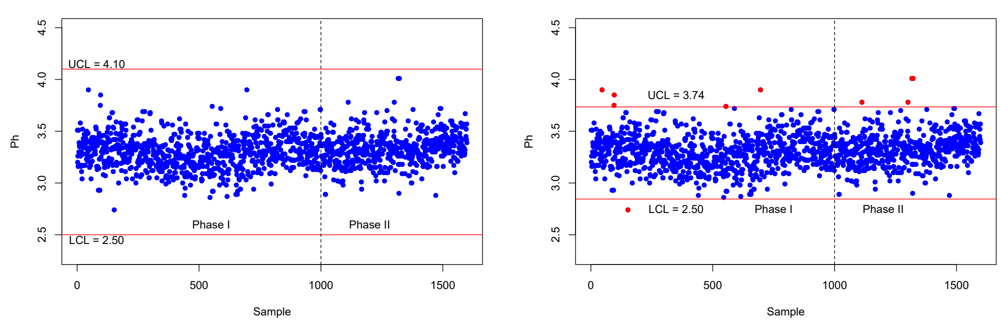

Figure 3. Control chart for monitoring the average pH of red wines from the Vinho Verde company produced from May 2004 to February 2007 with by the proposed (left) and he usual Shewhart (right) methods.

6. Final considerations

In this work, a monitoring method via bootstrap is proposed and evaluated in order to monitor the mean of symmetric data whose distribution belongs to the symmetric distribution class. This method comes as an alternative to Shewhart $\overline{X}$ chart when we want to monitor non-normal symmetric data, especially with heavy-tailed distribution data. The simulation study (illustrated with the Student-t and power-exponential distribution) shows that the proposed approach, for $\alpha = 0.0027$, provides in-control average run length between 340 and 380 samples and a good detection power, approaching one sample as n increases. Regarding the behavior of the control limits, for the proposed approach, they become closer to the mean when n increases.

In the context of light-tailed distribution, the proposed method presents good performance in-control average run length close to the desired value and good detection power). However, it is recommended the usual Shewhart $\overline{X}$ chart, because in addition to presenting a detection power comparable to the proposed method, it has a false alarm rate lower than that of the proposed method. It is worth nothing that the great advantage of using the proposed method, instead of the usual Shewhart method, is in situations in which the data distribution has a heavier tail than normal distribution, since the proposed method has a lower rate of false alarm (being very close to the nominal value) and excellent detection power, as seen in the simulation study and illustrated in the monitoring of the average pH of red wines. Finally, the proposed method is robust to dispersion index variation. As future work we highlight: (i) to analyze the effect of the parameter estimation in the proposed method, (ii) to consider a joint monitoring of the mean and the standard deviation for symmetric class data and, (iii) to propose an EWMA and CUSUM charts for the symmetric class to monitor small deviations from the mean.

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