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Statistical Process Control
Research Paper

Bootstrap control charts for the generalized Pareto distribution percentiles

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Abstract

Lifetime percentile is an important indicator of product reliability. Recently, numerous quality control charts have been built for the quantiles of different distributions. Because of the positive support and flexibility, the Pareto distribution is one of the useful distributions to model lifetime. But the statistical quality control for the Pareto percentiles has not been considered. The current work aims to establish quality control charts for the Pareto distribution percentiles. The least squared error, maximum likelihood and a modified moment method estimators are proposed for monitoring the Pareto distribution percentiles. However, the sampling distributions of percentile estimators are neither known nor bell shape. As a result, the well-known Shewhart-type control chart may not be appropriately applied to monitor the Pareto distribution percentiles. The bootstrap procedure and normality approximations are proposed to establish control charts. An intensive Monte Carlo simulation study is conducted to compare the performance among the proposed bootstrap and Shewhart-type control charts. The simulation study shows that the bootstrap control chart based on the maximum likelihood estimator outperforms the rest control charts considered. Finally, a numerical example is utilized to illustrate the application of the bootstrap control chart based on maximum likelihood estimator.

Keywords: Average run length · False alarm rate · Quality control chart · Parametric bootstrap · Percentile.

Mathematics Subject Classification: Primary 62F40 · secondary 62P30.

1. Introduction

As product lifetime is a key aspect metric in industry, certain standards for the quality of a product lifetime are often required to prevent faulty or inferior products from reaching the consumer (Aykroyd et al., 2019). The statistical quality control charts have been very useful tools to improve product lifetime quality as well as reliability. Therefore, researchers have developed percentile control charts for many different lifetime distributions recently. For example, Lio and Park (2008) studied control charts for Birnbaum-Saunders percentiles, Lio and Park (2010) explored control charts for the inverse Gaussian percentiles, Lio et al. (2014) developed quality control charts for the Burr type-X percentiles, Rezac et al. (2015) developed percentile control charts for the Burr type-XII distribution that has been

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published in the ChJS and Chiang et al. (2017) investigated the percentile control charts for the generalized exponential distributions.

Since Pickands (1975) introduced the Pareto distribution, many authors have studied the properties of the Pareto distribution and eventually developed the two-parameter generalized Pareto distribution that has the rate and shape parameters. Some of these authors include Hosking and Wallis (1987), Hüsler et al. (2011), Chen et al. (2017) and Salmasi and Yari (2017). Due to the flexibility of the two-parameter generalized Pareto distribution with positive support, the generalized Pareto distribution would have been useful for lifetime modeling. However, based on our best knowledge, no any research work has addressed the generalized Pareto distribution percentiles. Therefore, the goal of this study is to investigate the quality control of the generalized Pareto distribution percentiles.

When the exact sampling distribution of the parameter estimator is not available, the approximated sampling distributions, such as asymptotic normal or bootstrap sample distribution, would be used to inference the parameter concerned. However, the asymptotic normal distribution is usually for large sample size case or the bell-shape sampling distribution. For the quality control chart established based on small sample size, the commonly used Shewhart-type control chart that is based on normal distribution may not be appropriate because the sampling distribution of a percentile estimator is usually not either known nor near a bell shape one. Hence, the parametric bootstrap procedure has been proposed to approximate the sampling distribution of the percentile estimator such that the control chart could be built. For more information, readers may refer to the aforementioned works on percentile quality control charts. For a thorough introduction to the bootstrap method, see Gunter (1992), Efron and Tibshirani (1993) and Young (1994). One distinct advantage of the bootstrap method is it allows the establishment of control chart limits when the sampling distribution of an estimator is unknown. This paper uses this fact extensively while using the least square error (LSE), maximum likelihood (ML) and modified moment method (MMM) estimations, respectively. A minor disadvantage could be the computational time of the bootstrap method. Recently, with access to more powerful computers, the runtime can be reduced to a reasonable amount.

While studying in different areas, many authors confirmed the superiority of the bootstrap method to the Shewhart chart and shown significant characteristics of bootstrapping. To name a few, Nichols and Padgett (2005) found that a parametric bootstrap chart could detect an out of control process faster than a Shewhart-type chart. Lio and Park (2008), Lio and Park (2010), Lio et al. (2014), Rezac et al. (2015) and Chiang et al. (2017) showed that bootstrap charts based on the maximum likelihood estimate or the moment method estimate performed better than the Shewhart-type chart when monitoring the lifetime percentiles. The above discussions motivate the current investigation of the parametric bootstrap control charts based on the ML, MMM and LSE estimators, respectively, for the generalized Pareto percentiles. Then, all the proposed parametric bootstrap control charts and the Shewhart-type chart are compared using the computer simulation.

In order to create the control charts, Section 2 presents the three different estimation methods, which include the ML, MMM and LSE methods, for the unknown distribution parameters. The procedures of the Shewhart-type and parametric bootstrap charts are addressed in Section 3. After the control charts are developed based on the aforementioned four different estimation methods, the average of run lengths (ARLs), standard error of the ARL (SEARL) as well as the average of upper control limits (UCLs) and lower control limits (LCLs), and their respective standard deviations, are obtained through a simulation study and used to compare and determine which method is the best for monitoring the generalized Pareto percentiles in Section 4. In this same section, a numerical example is given for the illustration purpose. Finally, some remarks and suggestions is addressed in Section 5.

2. Parameter estimation

In this section, we introduce the two-parameter generalized Pareto distribution and three different estimation methods.

2.1 The generalized Pareto distribution

Let X be the random variable of the two-parameter generalized Pareto distribution that has the probability density function (PDF), cumulative distribution function (CDF) and percentile function respectively given as,

$$\begin{aligned} f(x;\alpha,\lambda) &= \alpha\lambda(1+x\lambda)^{-(\alpha+1)}, \quad x>0, \\ F(x;\alpha,\lambda) &= 1-(1+x\lambda)^{-\alpha}, \quad x>0 \\ Q(p;\alpha,\lambda) &= \frac{1}{\lambda}((1-p)^{-1/\alpha}-1), \quad 0<p<1, \end{aligned} \tag{1}$$

where $\lambda > 0$ is the rate parameter and $\alpha > 0$ is the shape parameter. Three estimation procedures for the unknown distribution parameters and percentiles is presented next.

2.2 The ML estimators

Let $X_1, \ldots, X_n$ be a random sample of size n from the generalized Pareto distribution given in Equation (1). The corresponding log-likelihood function is given as

$$l(\alpha,\lambda) = n\log(\alpha) + n\log(\lambda) - (\alpha+1)\sum_{i=1}^{n}\log(1+X_i\lambda).$$

Setting the partial derivative of $l(\alpha,\lambda)$ with respect to α and λ equal to zero, respectively, two normal equations are obtained as

$$\begin{aligned} \frac{n}{\alpha} &= \sum_{i=1}^{n}\log(1+X_i\lambda) \\ \frac{n}{\lambda} &= (\alpha+1)\sum_{i=1}^{n}\frac{X_i}{1+X_i\lambda}. \end{aligned} \tag{2}$$

The system of Equation (2) produces

$$\alpha = \frac{n}{\sum\limits_{i=1}^{n}\log(1+X_i\lambda)} \tag{3}$$

and

$$\frac{n}{\lambda} = \left(\frac{n}{\sum\limits_{i=1}^{n}\log(1+X_i\lambda)}+1\right)\sum_{i=1}^{n}\frac{X_i}{1+X_i\lambda}. \tag{4}$$

The solution of λ to Equation (4) could be obtained by the unit-root function, unit-root, in R and labeled by $\widehat{\lambda}_n$. Plugging $\widehat{\lambda}_n$ into Equation (3), the solution $\widehat{\alpha}_n$ is obtained. The solutions, $\widehat{\alpha}_n$ and $\widehat{\lambda}_n$, are called the ML estimates of α and λ, respectively. $\widehat{\alpha}_n$ and $\widehat{\lambda}_n$ may also be simultaneously obtained by optimization function `optim` of R. The ML estimate of the pth quantile can be stated as

$$\widehat{Q}_n(p;\widehat{\alpha}_n,\widehat{\lambda}_n) = \frac{1}{\widehat{\lambda}_n}\left((1-p)^{-1/\widehat{\alpha}_n - 1}\right), \quad 0<p<1.$$

However, the exact sampling distributions of $\widehat{\alpha}_n$, $\widehat{\lambda}_n$ and $\widehat{Q}_n(p;\widehat{\alpha}_n,\widehat{\lambda}_n)$ are unknown. Therefore, the exact quality control chart for $Q(p;\alpha,\lambda)$ cannot be established through $\widehat{Q}_n(p;\widehat{\alpha}_n,\widehat{\lambda}_n)$. It can be shown that $\sqrt{n}((\widehat{\alpha}_n,\widehat{\lambda}_n)-(\alpha,\lambda)) \to \mathrm{N}_2(\mathbf{0},\boldsymbol{I}^{-1}(\alpha,\lambda))$ where $\mathrm{N}_2$is the bivariate normal distribution with mean vector as the two-dimension zero vector, $\mathbf{0}$ and two by two variance covariance matrix as the inverse of the Fisher information matrix, $\boldsymbol{I}(\alpha,\lambda)$, given as

$$\boldsymbol{I}(\alpha,\lambda) = -\frac{1}{n}\begin{bmatrix} \mathrm{E}(\frac{\partial^2 l(\alpha,\lambda)}{\partial\alpha^2}) & \mathrm{E}(\frac{\partial l(\alpha,\lambda)}{\partial\alpha\partial\lambda}) \\ \mathrm{E}(\frac{\partial l(\alpha,\lambda)}{\partial\lambda\partial\alpha}) & \mathrm{E}(\frac{\partial^2 l(\alpha,\lambda)}{\partial\lambda^2}) \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\alpha^2} & \dfrac{1}{\lambda(\alpha+1)} \\ \dfrac{1}{\lambda(\alpha+1)} & \dfrac{1}{\lambda^2} - \dfrac{2}{\lambda^2(\alpha+2)} \end{bmatrix}.$$

More detail calculation procedures for the four entries of $\boldsymbol{I}(\alpha,\lambda)$ are as follows. The Fisher information matrix is presented as

$$\boldsymbol{I}(\alpha,\lambda) = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix},$$

where $I_{11}, I_{12}, I_{21}, I_{22}$ can be obtained through

$$\begin{aligned} I_{11} &= -\frac{1}{n}\mathrm{E}\left(\frac{\partial^2 l(\alpha,\lambda)}{\partial\alpha^2}\right) = \frac{1}{\alpha^2} \\ I_{12} = I_{21} &= -\frac{1}{n}\mathrm{E}\left(\frac{\partial l(\alpha,\lambda)}{\partial\alpha\partial\lambda}\right) = -\frac{1}{n}\mathrm{E}\left(-\sum_{i=1}^{n}\frac{x_i}{1+x_i\lambda}\right) \\ &= \frac{1}{n}\sum_{i=1}^{n}\int_0^\infty \frac{x}{1+x\lambda}f(x)\mathrm{d}x = \alpha\lambda\int_0^\infty x(1+x\lambda)^{-\alpha-2}\mathrm{d}x = \frac{1}{\lambda(\alpha+1)}, \\ I_{22} &= -\frac{1}{n}\mathrm{E}\left(\frac{\partial^2 l(\alpha,\lambda)}{\partial\lambda^2}\right) = -\frac{1}{n}\mathrm{E}\left(-\frac{n}{\lambda^2} + (\alpha+1)\sum_{i=1}^{n}\frac{{X_i}^2}{(1+x_i\lambda)^2}\right) = \frac{1}{\lambda^2} - \frac{2}{\lambda^2(\alpha+2)} \end{aligned}$$

It can be shown that

$$\frac{\widehat{Q}_n(p;\widehat{\alpha}_n,\widehat{\lambda}_n) - Q(p;\alpha,\lambda)}{\sigma_{p,n}} \to \mathrm{N}(0,1), \quad 0<p<1,$$

where $\sigma^2_{p,n} = (1/n)\nabla Q(p;\alpha,\lambda)^\top \boldsymbol{I}^{-1}(\alpha,\lambda)\nabla Q(p;\alpha,\lambda)$, for $0<p<1$, and $\nabla Q(p;\alpha,\lambda)$ is the gradient of $Q(p,\alpha,\lambda)$ with respect to α and λ. Thus, a Shewhart chart can be constructed using the asymptotic normal distribution to monitor the generalized Pareto percentile.

2.3 The MMM estimators

Given n sample observations, $x_1, \ldots, x_n$, from the generalized Pareto distribution. In order to find the moment method estimates of α and λ, let the first order sample moment about zero be equal to the population mean and the second order sample moment about zero be equal to the population second moment about zero. Then, two required equations for moment method estimates can be expressed as

$$\mathrm{E}(X) = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad \mathrm{E}(X^2) = \frac{1}{n}\sum_{i=1}^{n} x_i^2, \tag{5}$$

where X is the generalized Pareto distribution random variable. However, the solutions to the system of Equation (5) are difficult to obtain. Also the α solution is restricted to $\alpha > 2$ to ensure $\mathrm{E}(X^2)$ finite. Hence, a modified moment method is needed so that the solutions are easier to solve and there is no restriction on α.

Let $U = \alpha \log(1 + \lambda X)$, then it can be easily shown that U has exponential distribution with mean equal 1. Let $X_{(1)} < \cdots < X_{(n)}$ be the ordered statistic of $X_1, \ldots, X_n$. Then, $U_{(1)} < \cdots < U_{(n)}$ is the ordered statistic of $U_i = \alpha \log(1 + \lambda X_i)$ for $i = 1, \ldots, n$. Denote $Y_1 = nU_{(1)}, \ldots, Y_n = U_{(n)} - U_{(n-1)}$ or alternatively, $Y_1 = n\alpha \log(1 + \lambda X_{(1)}), Y_2 = (n - 1)\alpha(\log(1 + \lambda X_{(2)}) - \log(1 + \lambda X_{(1)})), \ldots, Y_n = \alpha(\log(1 + \lambda X_{(n)}) - \log(1 + \lambda X_{(n-1)}))$. It can be shown that $Y_1, \ldots, Y_n$ are random sample from the exponential distribution with mean equal to one. Let $g(\lambda) = 2\sum_{i=1}^{n-1}(\log(T_n) - \log(T_i))$, where $T_i = \sum_{j=1}^{i} Y_j/\alpha$. It can be shown that $g(\lambda)$ has a chi-square distribution with degree of freedom $2n-2$ and αT_n has a gamma distribution $G(1, n)$. Following Wang (2008) and Rezac et al. (2015), λ can be estimated by the unique solution of $g(\tilde{\lambda}) = 2(n - 2)$ and α can be estimated by $\tilde{\alpha} = (n - 1)/T_n$. The estimates, $\tilde{\lambda}_n$ and $\tilde{\alpha}_n$, are called MMM estimations of λ and α, respectively. Then, the MMM estimator of the generalized Pareto percentile $Q(p, \alpha, \lambda)$ based on the MMM estimators, $\tilde{\alpha}_n$ and $\tilde{\lambda}_n$, is defined as $\tilde{Q}_n(p; \tilde{\alpha}_n, \tilde{\lambda}_n) = (1/\tilde{\lambda}_n)\left((1 - p)^{-1/\tilde{\alpha}_n} - 1\right)$, for $0 < p < 1$. However, the exact sampling distributions of $\tilde{\alpha}_n$, $\tilde{\lambda}_n$ and $\tilde{Q}_n(p; \tilde{\alpha}_n, \tilde{\lambda}_n)$ are unknown.

2.4 The LSE estimators

The LSE estimators of the generalized Pareto distribution parameters are obtained by minimizing the following sum of squares with respect to α and λ,

$$\sum_{i=1}^{n}\left(F(X_{(i)}; \alpha, \lambda) - \frac{i}{n+1}\right)^2,$$

where $F(X_{(i)}; \alpha, \lambda) = 1 - (1 + X_{(i)}\lambda)^{-\alpha}$ for $i = 1, \ldots, n$. The solutions of α and λ can be obtained simultaneously by optimization function, optim, in R and are labeled by $\overline{\alpha}_n$ and $\overline{\lambda}_n$, respectively. Then, the LSE estimator of $Q(p; \alpha, \lambda)$ is defined as $\overline{Q}_n(p; \overline{\alpha}_n, \overline{\lambda}_n) = (1/\overline{\lambda}_n)((1 - p)^{-1/\overline{\alpha}_n} - 1)$, for $0 < p < 1$. Again, the exact sampling distributions of $\overline{\alpha}_n$,$\overline{\lambda}_n$ and $\overline{Q}_n(p; \overline{\alpha}_n, \overline{\lambda}_n)$, respectively, are unknown.

3. Statistical control charts

In this section, we perform the Shewhart-type and parametric bootstrap charts.

3.1 Assumptions

In Phase I, there are several assumptions, which are that the k in-control subgroup samples of size m are randomly collected from the generalized Pareto PDF of Equation (1) for the control chart setting. Let $n = m \times k$ denote the total sample size used in Phase I and $\widehat{\alpha}_n$ and $\widehat{\lambda}_n$ be the ML estimates of α and λ, respectively. The process for creating the Shewhart-type and parametric bootstrap charts is illustrated in the following sections.

3.2 The Shewhart-type charts

Using the ML estimation procedure described in Section 2.2, the ML estimate of the 100pth percentile for $0 < p < 1$ based on each subgroup sample of size m from the Phase I process is $\widehat{Q}_m(p; \widehat{\alpha}_m, \widehat{\lambda}_m) = (1/\widehat{\lambda}_m)((1-p)^{-1/\widehat{\alpha}_m} - 1)$ where $(\widehat{\alpha}_m, \widehat{\lambda}_m)$ is the ML estimates of (α, λ). Then, the Shewhart-type chart for monitoring the 100pth percentile, $Q(p; \alpha, \lambda)$, by using $\widehat{Q}_m(p; \widehat{\alpha}_m, \widehat{\lambda}_m)$ for $0 < p < 1$ can be constructed with the steps:

(1) Using all n sample observations from Phase I in-control process, the ML estimates of α and λ were obtained above. Then the asymptotic standard error of $\widehat{Q}_m(p; \widehat{\alpha}_m, \widehat{\lambda}_m)$ can be estimated by

$$\widehat{\sigma}_{\widehat{Q}_m} = \sqrt{\frac{1}{m} \nabla Q^\top(p; \widehat{\alpha}_n, \widehat{\lambda}_n) \widehat{\boldsymbol{I}}_n^{-1}(\widehat{\alpha}_n, \widehat{\lambda}_n) \nabla Q(p; \widehat{\alpha}_n, \widehat{\lambda}_n)}.$$

(2) For the jth subgroup sample of size m, the ML estimates of α, λ and $Q(p; \alpha, \lambda)$ are found by using the procedure of Section 2.2 and denoted by $\widehat{\alpha}_m^j$, $\widehat{\lambda}_m^j$ and $\widehat{Q}_m^j(p; \widehat{\alpha}_m^j, \widehat{\lambda}_m^j)$, respectively, for $j = 1, \ldots, k$. The sample mean, $\bar{\widehat{Q}}_m(p)$, of $\widehat{Q}_m^j(p; \widehat{\alpha}_m^j, \widehat{\lambda}_m^j)$ for $j = 1, \ldots, k$ is obtained as

$$\bar{\widehat{Q}}_m(p) = \frac{1}{k} \sum_{j=1}^{k} \widehat{Q}_m^j(p; \widehat{\alpha}_m^j, \widehat{\lambda}_m^j).$$

(3) The control limits of the Shewhart-type chart are given as

$$\mathrm{LCL}_{\mathrm{SH}} = \bar{\widehat{Q}}_m(p) - z_{(1-\gamma/2)} \widehat{\sigma}_{\widehat{Q}_m}, \quad \mathrm{UCL}_{\mathrm{SH}} = \bar{\widehat{Q}}_m(p) + z_{(1-\gamma/2)} \widehat{\sigma}_{\widehat{Q}_m},$$

where $\bar{\widehat{Q}}_m(p)$ is the center line (CL), $z_{1-\gamma/2}$ satisfies $\Phi(z_{1-\gamma/2}) = 1 - \gamma/2$ with $0 < \gamma < 1$, Φ is the standard normal CDF and γ is the false alarm rate (FAR).

After the control limits of the Shewhart-type chart are determined, future samples of size m (Phase II samples) are drawn from the generalized Pareto process to compute the plot statistic $\widehat{Q}_m(p; \widehat{\alpha}_m, \widehat{\lambda}_m)$. If $\widehat{Q}_m(p; \widehat{\alpha}_m, \widehat{\lambda}_m)$ is between the control limits found above, then the process is assumed to be in control. If not, signal that the process is out-of-control.

3.3 Parametric bootstrap charts

The parametric bootstrap chart based on the ML estimation method is constructed as

(1) Using all n observations collected during the Phase I in-control process, the ML estimates, $\widehat{\alpha}_n$ and $\widehat{\lambda}_n$, of α and λ were obtained above.
(2) Generate m parametric bootstrap observations from the generalized Pareto distribution given in Equation (1), with $\alpha = \widehat{\alpha}_n$ and $\lambda = \widehat{\lambda}_n$. Denote the parametric bootstrap observations by $x_1^*, \ldots, x_m^*$.

(3) Find the ML estimates of α and λ using $x_1^*, \ldots, x_m^*$ from Step 2. The obtained ML estimates of α and λ are labeled by $\widehat{\alpha}_m^*$ and $\widehat{\lambda}_m^*$, respectively.

(4) Find the bootstrap estimate of the 100pth percentile, denoted $\widehat{Q}_m^*(p; \widehat{\alpha}_m^*, \widehat{\lambda}_m^*)$ by plugging $\widehat{\alpha}_m^*$ and $\widehat{\lambda}_m^*$ into the quantile function, $Q(p; \alpha, \lambda)$, that is,

$$\widehat{Q}_m^*(p; \widehat{\alpha}_m^*, \widehat{\lambda}_m^*) = Q(p; \widehat{\alpha}_m^*, \widehat{\lambda}_m^*) = \frac{1}{\widehat{\lambda}_m^*}((1-p)^{-1/\widehat{\alpha}_m^*} - 1).$$

(5) Repeat Steps 2 through 4 M times to obtain a size M bootstrap sample, $\widehat{Q}_{m,j}^*(p; \widehat{\alpha}_{m,j}^*, \widehat{\lambda}_{m,j}^*), j = 1, \ldots, M$, where M is a given large positive integer.

(6) Given a FAR,γ, find the $(\gamma/2)$th and (1-$\gamma/2$)th empirical quantiles of the bootstrap sample from Step 5 as the LCL and UCL, respectively, where the empirical quantiles can be obtained by using `R` quantile function. The CL is given as

$$\bar{\widehat{Q}}_m^*(p) = \frac{1}{M} \sum_{j=1}^{M} \widehat{Q}_{m,j}^*(p; \widehat{\alpha}_{m,j}^*, \widehat{\lambda}_{m,j}^*).$$

The LCL and UCL developed above and the plot statistic, $\widehat{Q}_m^*(p; \widehat{\alpha}_m^*, \widehat{\lambda}_m^*)$ is called the ML bootstrap chart. Following the same steps established in this section and replacing $\widehat{\alpha}(\widehat{\alpha}^*)$ and $\widehat{\lambda}(\widehat{\lambda}^*)$ by $\tilde{\alpha}(\tilde{\alpha}^*)$ and $\tilde{\lambda}(\tilde{\lambda}^*)$, respectively. Then, the corresponding MMM bootstrap chart is obtained. Similarly, replacing $\widehat{\alpha}(\widehat{\alpha}^*)$ and $\widehat{\lambda}(\widehat{\lambda}^*)$ by $\overline{\alpha}(\overline{\alpha}^*)$ and $\overline{\lambda}(\overline{\lambda}^*)$, respectively. Then, the corresponding LSE bootstrap chart is developed.

4. Numerical studies

In this section, based on the aforementioned four different estimation methods, we obtain the ARL, SEARL, UCL, LCL, and their respective standard deviations, through a simulation study and used to compare and determine which method is the best for monitoring the generalized Pareto percentiles. We close this section with a numerical example to show potential applications.

4.1 Simulation scenario

To compare the performance among the proposed generalized Pareto distribution quantile control charts for monitoring lower quantiles below median, a Monte Carlo simulation study was executed using `R`, a programming language and environment originally developed by Ihaka and Gentleman (1996). The `R` code is available from the authors on request.

The performance quality of the control charts was based on the ARL and its SEARL. The average LCL and average UCL and their corresponding standard errors were also recorded for each method discussed in Section 2. From a practical standpoint, very few samples are available for lifetime testing in industry for quality control because the lifetime test is destructive and expensive. Hence, in the simulation, only sample sizes of $m = 4, 5$ and 6 with $k = 20$ subgroups were collected randomly. This simulation also considered a variety of false alarm rates (FARs), specifically, 0.1, 0.01, 0.0027, and 0.002. The control limits of 100pth percentiles, where $p = 0.01, 0.05, 0.10$ and 0.25, were found using the empirical distribution of $M = 10,000$ bootstrap observations for bootstrap control charts. The simulation process was repeated 10,000 times to find an accurate estimation of the ARL, SEARL, average of LCL, average of UCL and their respective standard errors running a self developed R program through the hp laptop with window 10. It took about 16.5

hours to run each one submission of R program for monitoring one percentile to product $10,000$ LCLs, UCLs and run lengths for four control charts with FAR $= 0.1, 0.01, 0.0027$ and 0.002, respectively. The ARL, average LCL and average UCL are the average of $10,000$ run lengths, LCLs and UCLs, respectively. The SEARL and standard deviations of the average of LCL and UCL are calculated by using standard deviations of $10,000$ run lengths, LCLs and UCLs divided by squared root of $10,000$, respectively.

4.2 Simulation results

In Tables 1 through 4, the ARLs and SEARLs are compared. An appropriate control chart has ARL near 1/FAR that is also known as the nominal ARL. The simulated ARLs and SEARLs for the Shewhart-type chart are shown in Table 1. The Shewhart-type chart overestimates the nominal ARL for FAR=0.1 and underestimates for FAR=$0.01, 0.0027$, and 0.002. That indicates overall narrow control limits except the case of FAR $= 0.1$. Table 2 shows the simulated ARLs and SEARLs for the LSE bootstrap chart. This process highly overestimates the nominal ARL. Table 3 shows the simulated ARLs and standard deviations for the MMM bootstrap chart. This chart is simply inconsistent. For FAR=0.1, it does fairly well across the percentiles tested. However, for smaller FAR, it underestimates the nominal ARL for smaller percentiles and overestimates for larger percentiles. Finally, Table 4 shows the simulated ARLs and SEARLs for the ML bootstrap chart. The ARLs in this chart stay close to the nominal ARL with small SEARL relative to the corresponding ARL across all percentiles and sample sizes used. All SEARLs shown in Tables 1 through 4 are very small compared with their respective ARLs. In Tables 5 through 8, the averages of LCLs and UCLs of each chart are compared. In Table 5, notice that the Shewhart-type chart has a negative average lower bound. Since the charts are used to monitor lifetime data, a negative lower bound implies that the normal approximation is not appropriate and the Shewhart-type chart is not appropriate for detecting a low percentile deteriorate. Also note that some of the average LCLs shown in Table 7 for the MMM bootstrap charts are set at 0+. Actually those numbers are very small positive. Again, the MMM bootstrap charts are not appropriate to use particularly monitoring a low percentile deteriorate. The calculated standard errors of average LCLs and UCLs for each control chart from $10,000$ simulation runs are displayed in Tables 9 through 12. Some calculated standard errors for the LCLs and UCLs of the MMM bootstrap chart shown in Table 11 are 0+ that are actually very small positive numbers. Tables 9 through 12 show all standard errors are very small. As the Shewhart-type chart, the LSE bootstrap chart, and the MMM bootstrap chart have all been eliminated from consideration, the only chart left is the ML bootstrap chart. This chart's ARL stays close to the nominal ARL (Table 4) with small SEARL relative to the corresponding ARL, its LCLs and UCLs are reliable (Table 8), and the standard error for the averages of LCLs and UCLs are small. As a result, the ML bootstrap chart is assessed for monitoring the out of control.

Out of control testing analyzes how quickly the ML bootstrap chart detects a downward shift in the distribution percentiles. This type of downward shift indicates a product's lifetime is shortening. Looking at the Pareto quantile function, it is clear that as the α and (or) λ increase, the quantile function decreases. As a result, when running the out of control testing on ML bootstrap chart, the parameters α and (or) λ were shifted upwards. The results of this test are mainly based on the ARL and SEARL. To calculate ARL and SEARL for each out of control setting, the simulation study were conducted $10,000$ runs and each run with $10,000$ bootstrap sample observations. Table 13 through Table 15 display the simulation results. In Table 13, both α and λ shifted upwards. Let α_0 and λ_0 be the in-control parameter inputs and increase the values of α_0 and λ_0 to α_1 and λ_1. In Table 14, λ_0 from the in-control process is fixed and α_0 from the in-control process is increased to α_1. In Table 15, α_0 from the in-control process is fixed and λ_0 from the in-control process is increased to λ_1. In viewing of Table 13 through Table 15, it can been seen that all ARLs are relatively small compared to the nominal ARL and all SEARLs are very small, too. These results confirm that the ML bootstrap control chart is reliable for monitoring generalized Pareto percentiles.

Table 1. Shewhart-type in-control ARL estimates and corresponding standard deviations for generalized Pareto percentiles with $\alpha = 2.5$ and $\lambda = 1.0$.

Parameters	$n = 4$		$n = 5$		$n = 6$	
	ARL	SEARL	ARL	SEARL	ARL	SEARL
	$\gamma_0 = 0.1$ (FAR)		$1/\gamma_0 = 10$			
$p = 0.01$	15.4038	0.3053	16.1360	0.2922	15.7627	0.2674
$p = 0.05$	15.2520	0.3026	15.9680	0.2886	15.6154	0.2619
$p = 0.10$	15.2592	0.3036	15.6528	0.2785	15.3232	0.2564
$p = 0.25$	14.6227	0.2501	15.1762	0.2680	14.6056	0.2441
	$\gamma_0 = 0.01$ (FAR)		$1/\gamma_0 = 100$			
$p = 0.01$	37.9077	0.8082	43.0204	0.8511	45.0765	0.8593
$p = 0.05$	37.4736	0.7973	42.5070	0.8302	44.4339	0.8347
$p = 0.10$	37.0809	0.7489	41.5448	0.7988	43.4403	0.8132
$p = 0.25$	35.0311	0.6345	39.4974	0.7754	40.9858	0.7740
	$\gamma_0 = 0.0027$ (FAR)		$1/\gamma_0 = 370.37$			
$p = 0.01$	55.3873	1.1429	65.0647	1.3018	70.2976	1.3643
$p = 0.05$	54.7378	1.1404	64.7746	1.3200	70.2678	1.3980
$p = 0.10$	53.9932	1.1237	63.4313	1.2958	68.8031	1.3463
$p = 0.25$	50.3986	0.9611	59.6773	1.2202	64.3715	1.2834
	$\gamma_0 = 0.002$ (FAR)		$1/\gamma_0 = 500$			
$p = 0.01$	59.9459	1.2528	70.5459	1.4171	78.2119	1.5724
$p = 0.05$	59.0701	1.2362	70.0550	1.4683	77.8436	1.5785
$p = 0.10$	58.2528	1.2071	68.6909	1.4198	76.4253	1.5344
$p = 0.25$	54.0512	1.0232	64.5977	1.3066	71.1732	1.4163

Table 2. LSE bootstrap in-control ARLs estimates and corresponding standard deviations for the generalized Pareto percentiles with $\alpha = 2.5$ and $\lambda = 1.0$.

Parameters	$n = 4$		$n = 5$		$n = 6$	
	ARL	SEARL	ARL	SEARL	ARL	SEARL
	$\gamma_0 = 0.1$ (FAR)		$1/\gamma_0 = 10$			
$p = 0.01$	12.740	0.1918	12.7904	0.1954	12.7874	0.1900
$p = 0.05$	12.7783	0.1901	12.7938	0.1982	12.7571	0.1914
$p = 0.10$	12.7063	0.1947	12.9092	0.1978	12.8133	0.1892
$p = 0.25$	13.1900	0.2027	13.1257	0.2076	13.2774	0.2036
	$\gamma_0 = 0.01$ (FAR)		$1/\gamma_0 = 100$			
$p = 0.01$	179.819	3.4544	186.3573	3.6381	189.9276	3.8459
$p = 0.05$	180.4755	3.4162	185.7970	3.7856	195.4479	4.1130
$p = 0.10$	180.9999	3.4464	187.6843	3.7808	190.5604	3.9170
$p = 0.25$	184.2501	3.6159	188.6343	3.7525	194.9903	4.0870
	$\gamma_0 = 0.0027$ (FAR)		$1/\gamma_0 = 370.37$			
$p = 0.01$	788.459	17.7730	812.0935	18.4439	844.5480	19.1808
$p = 0.05$	788.6242	17.1870	821.3853	18.4914	870.5871	19.7251
$p = 0.10$	771.9857	16.2963	819.5059	17.9659	852.3299	19.8192
$p = 0.25$	796.4193	17.0323	807.9862	17.5517	872.3674	21.6409
	$\gamma_0 = 0.002$ (FAR)		$1/\gamma_0 = 500$			
$p = 0.01$	1120.381	24.9256	1165.7726	29.7523	1234.9286	30.7898
$p = 0.05$	1127.6833	24.9995	1172.5420	28.1306	1245.6503	29.3309
$p = 0.10$	1101.3921	23.7495	1193.0949	29.3921	1221.7993	28.5823
$p = 0.25$	1118.6962	24.9399	1148.6458	27.4038	1272.1061	31.8478

Table 3. MMM bootstrap in-control ARL estimates and corresponding standard deviations for the generalized Pareto percentiles with $\alpha = 2.5$ and $\lambda = 1.0$.

Parameters	$n = 4$		$n = 5$		$n = 6$	
	ARL	SEARL	ARL	SEARL	ARL	SEARL
	$\gamma_0 = 0.1$ (FAR)		$1/\gamma_0 = 10$			
$p = 0.01$	9.6570	0.1335	9.7628	0.1373	9.6871	0.1332
$p = 0.05$	9.3388	0.1352	11.6867	0.1609	10.9639	0.1515
$p = 0.10$	8.7744	0.1272	13.2327	0.2123	10.3124	0.1400
$p = 0.25$	9.6432	0.1406	12.4179	0.2023	10.7797	0.1503
	$\gamma_0 = 0.01$ (FAR)		$1/\gamma_0 = 100$			
$p = 0.01$	49.7678	0.7441	55.1232	0.8122	58.5411	0.8671
$p = 0.05$	89.8256	1.2590	116.3314	1.5610	88.5108	1.2978
$p = 0.10$	109.3266	1.9285	102.2867	1.2401	114.7982	1.6952
$p = 0.25$	102.24451	1.8508	113.6496	1.4729	137.2954	2.4680
	$\gamma_0 = 0.0027$ (FAR)		$1/\gamma_0 = 370.37$			
$p = 0.01$	60.9588	0.8813	68.8012	0.9745	74.45920	1.0695
$p = 0.05$	147.0226	2.2652	191.4860	3.0285	186.7040	2.4625
$p = 0.10$	445.3300	8.0744	383.9819	7.0894	427.8512	6.3043
$p = 0.25$	377.1021	7.2660	452.4614	7.5417	467.0340	7.4687
	$\gamma_0 = 0.002$ (FAR)		$1/\gamma_0 = 500$			
$p = 0.01$	62.1741	0.8982	70.4454	0.9950	76.7008	1.1057
$p = 0.05$	156.1286	2.3482	201.7414	2.9981	202.7630	2.5971
$p = 0.10$	561.8492	10.0323	720.5037	14.6875	564.9076	9.8149
$p = 0.25$	589.4910	9.4193	606.7636	9.4168	576.9963	10.5925

Table 4. ML bootstrap in-control ARL estimates and corresponding standard deviations for generalized Pareto percentiles with $\alpha = 2.5$ and $\lambda = 1.0$.

Parameters	$n = 4$		$n = 5$		$n = 6$	
	ARL	SEARL	ARL	SEARL	ARL	SEARL
	$\gamma_0 = 0.1$ (FAR)		$1/\gamma_0 = 10$			
$p = 0.01$	9.3209	0.1309	9.0851	0.1250	9.2029	0.1266
$p = 0.05$	9.2294	0.1313	9.2691	0.1294	9.2289	0.1295
$p = 0.10$	9.2964	0.1316	9.1935	0.1302	9.3212	0.1292
$p = 0.25$	9.1746	0.1280	9.3039	0.1285	9.2309	0.1278
	$\gamma_0 = 0.01$ (FAR)		$1/\gamma_0 = 100$			
$p = 0.01$	91.6366	1.6801	89.7949	1.5565	88.8241	1.4242
$p = 0.05$	90.2183	1.6010	90.1718	1.5683	88.2176	1.4181
$p = 0.10$	90.3458	1.6343	91.3064	1.5865	90.5653	1.4959
$p = 0.25$	93.2211	1.6978	90.6854	1.5839	87.6807	1.4678
	$\gamma_0 = 0.0027$ (FAR)		$1/\gamma_0 = 370.37$			
$p = 0.01$	343.0466	7.0766	336.3652	6.4181	335.8107	6.2239
$p = 0.05$	339.5141	6.7849	333.6685	6.5899	328.6546	6.2053
$p = 0.10$	337.7009	6.8637	339.3910	6.4750	340.7111	6.4211
$p = 0.25$	355.4715	7.5188	340.8538	6.7502	330.1604	6.1004
	$\gamma_0 = 0.002$ (FAR)		$1/\gamma_0 = 500$			
$p = 0.01$	472.3910	10.1697	456.5624	9.1939	449.1549	8.4255
$p = 0.05$	470.6944	10.0895	452.7038	9.4391	445.0275	8.8036
$p = 0.10$	459.5464	9.7480	464.8904	9.3621	458.1780	8.9725
$p = 0.25$	488.0070	10.7023	464.8904	10.4099	450.7093	8.8848

Table 5. Shewhart-type in-control LCL and UCL for generalized Pareto percentiles with $\alpha = 2.5$ and $\lambda = 1.0$.

Parameters	$n = 4$		$n = 5$		$n = 6$	
	LCL	UCL	LCL	UCL	LCL	UCL
	$\gamma_0 = 0.1$ (FAR)					
$p = 0.01$	-0.0008	0.0104	-0.0003	0.0097	0.0001	0.0092
$p = 0.05$	-0.0037	0.0533	-0.0013	0.0495	0.0005	0.0469
$p = 0.10$	-0.0073	0.1096	-0.0022	0.1019	0.0016	0.0966
$p = 0.25$	-0.0152	0.3002	-0.0018	0.2800	0.0085	0.2658
	$\gamma_0 = 0.01$ (FAR)					
$p = 0.01$	-0.0039	0.0136	-0.0031	0.0125	-0.0025	0.0118
$p = 0.05$	-0.0199	0.0694	-0.0157	0.0639	-0.0126	0.0601
$p = 0.10$	-0.0403	0.1427	-0.0316	0.1314	-0.0253	0.1235
$p = 0.25$	-0.1044	0.3894	-0.0814	0.3597	-0.0643	0.3386
	$\gamma_0 = 0.0027$ (FAR)					
$p = 0.01$	-0.0054	0.0151	-0.0044	0.0138	-0.0037	0.0129
$p = 0.05$	-0.0272	0.0768	-0.0222	0.0705	-0.0186	0.0660
$p = 0.10$	-0.0554	0.1578	-0.0451	0.1448	-0.0376	0.1357
$p = 0.25$	-0.1451	0.4301	-0.1177	0.3961	-0.0974	0.3718
	$\gamma_0 = 0.002$ (FAR)					
$p = 0.01$	-0.0057	0.0154	-0.0047	0.0141	-0.0039	0.0132
$p = 0.05$	-0.0288	0.0784	-0.0236	0.0719	-0.0199	0.0673
$p = 0.10$	-0.0586	0.1610	-0.0479	0.1477	-0.0402	0.1383
$p = 0.25$	-0.1538	0.4387	-0.1254	0.4038	-0.1045	0.3788

Table 6. LSE bootstrap in-control LCL and UCL for the generalized Pareto percentiles with $\alpha = 2.5$ and $\lambda = 1.0$.

Parameters	$n = 4$		$n = 5$		$n = 6$	
	LCL	UCL	LCL	UCL	LCL	UCL
	$\gamma_0 = 0.1$ (FAR)					
$p = 0.01$	0.0011	0.0104	0.0012	0.0094	0.0014	0.0088
$p = 0.05$	0.0057	0.0538	0.0064	0.0486	0.0070	0.0454
$p = 0.10$	0.0117	0.1127	0.0133	0.1021	0.0146	0.0948
$p = 0.25$	0.0313	0.3344	0.0378	0.3008	0.0417	0.2779
	$\gamma_0 = 0.01$ (FAR)					
$p = 0.01$	0.0005	0.0221	0.0006	0.0183	0.0007	0.0160
$p = 0.05$	0.0025	0.1142	0.0031	0.0940	0.0038	0.0826
$p = 0.10$	0.0051	0.2376	0.0065	0.1970	0.0078	0.1714
$p = 0.25$	0.0142	0.6872	0.0181	0.5679	0.0219	0.4937
	$\gamma_0 = 0.0027$ (FAR)					
$p = 0.01$	0.0003	0.0329	0.0004	0.0257	0.0005	0.0215
$p = 0.05$	0.0016	0.1697	0.0022	0.1314	0.0027	0.1115
$p = 0.10$	0.0033	0.3523	0.0044	0.2756	0.0056	0.2307
$p = 0.25$	0.0091	1.0105	0.0124	0.7889	0.0158	0.6608
	$\gamma_0 = 0.002$ (FAR)					
$p = 0.01$	0.0003	0.0354	0.0004	0.0274	0.0005	0.0227
$p = 0.05$	0.0014	0.1825	0.0019	0.1399	0.0025	0.1178
$p = 0.10$	0.0029	0.3791	0.0040	0.2932	0.0052	0.2437
$p = 0.25$	0.0081	1.0857	0.0112	0.8393	0.0145	0.6969

Table 7. MMM bootstrap in-control LCL and UCL for the generalized Pareto percentiles with $\alpha = 2.5$ and $\lambda = 1.0$.

Parameters	$n=4$		$n=5$		$n=6$	
	LCL	UCL	LCL	UCL	LCL	UCL
	$\gamma_0 = 0.1$ (FAR)					
$p = 0.01$	0.0005	0.0140	0.0006	0.0122	0.0007	0.0112
$p = 0.05$	0.0044	0.0577	0.0036	0.0633	0.0044	0.0577
$p = 0.10$	0.0070	0.1491	0.0084	0.1307	0.0098	0.1186
$p = 0.25$	0.0248	0.4184	0.0284	0.3636	0.0323	0.3306
	$\gamma_0 = 0.01$ (FAR)					
$p = 0.01$	0.0000+	0.0254	0.0000+	0.0210	0.0000+	0.0183
$p = 0.05$	0.0004	0.0961	0.0002	0.1096	0.0004	0.0961
$p = 0.10$	0.0010	0.2700	0.0016	0.2247	0.0022	0.1943
$p = 0.25$	0.0051	0.7561	0.0069	0.6235	0.0091	0.5396
	$\gamma_0 = 0.0027$ (FAR)					
$p = 0.01$	0.0000+	0.0336	0.0000+	0.0269	0.0000+	0.0230
$p = 0.05$	0.0000+	0.1195	0.0000+	0.1391	0.0000+	0.1195
$p = 0.10$	0.0002	0.3583	0.0005	0.2878	0.0008	0.2443
$p = 0.25$	0.0019	1.0049	0.0029	0.8000	0.0042	0.6782
	$\gamma_0 = 0.002$ (FAR)					
$p = 0.01$	0.0000+	0.0353	0.0000+	0.0282	0.0000+	0.0239
$p = 0.05$	0.0000+	0.1240	0.0000+	0.1456	0.0000+	0.1241
$p = 0.10$	0.0001	0.3773	0.0004	0.3005	0.0006	0.2541
$p = 0.25$	0.0015	1.0573	0.0023	0.8373	0.0034	0.7046

Table 8. ML bootstrap in-control upper and lower control limits for generalized Pareto percentiles with $\alpha = 2.5$ and $\lambda = 1.0$.

Parameters	$n=4$		$n=5$		$n=6$	
	LCL	UCL	LCL	UCL	LCL	UCL
	$\gamma_0 = 0.1$ (FAR)					
$p = 0.01$	0.0011	0.0116	0.0012	0.0106	0.0013	0.0099
$p = 0.05$	0.0057	0.0595	0.0064	0.0546	0.0070	0.0510
$p = 0.10$	0.0120	0.1225	0.0134	0.1127	0.0146	0.1055
$p = 0.25$	0.0346	0.3408	0.0388	0.3166	0.0424	0.2918
	$\gamma_0 = 0.01$ (FAR)					
$p = 0.01$	0.0006	0.0210	0.0007	0.0181	0.0008	0.0161
$p = 0.05$	0.0030	0.1072	0.0036	0.926	0.0041	0.0828
$p = 0.10$	0.0062	0.2207	0.0075	0.1913	0.0085	0.1713
$p = 0.25$	0.0176	0.6149	0.0214	0.5284	0.0245	0.4729
	$\gamma_0 = 0.0027$ (FAR)					
$p = 0.01$	0.0004	0.0278	0.0005	0.0232	0.0006	0.0202
$p = 0.05$	0.0022	0.1418	0.0028	0.1185	0.0032	0.1039
$p = 0.10$	0.0045	0.2918	0.0058	0.2452	0.0068	0.2148
$p = 0.25$	0.0126	0.8144	0.0163	0.6761	0.0192	0.5923
	$\gamma_0 = 0.002$ (FAR)					
$p = 0.01$	0.0004	0.0292	0.0005	0.0242	0.0006	0.0211
$p = 0.05$	0.0020	0.1492	0.0026	0.1239	0.0031	0.1081
$p = 0.10$	0.0041	0.3068	0.0054	0.2564	0.0064	0.2238
$p = 0.25$	0.0116	0.8567	0.0151	0.7071	0.0181	0.6168

Table 9. The Shewhart-type chart standard errors for the UCL and LCL.

Parameters	$n=4$		$n=5$		$n=6$	
	LCL	UCL	LCL	UCL	LCL	UCL
	$\gamma_0 = 0.1$ (FAR)					
$p = 0.01$	7.39×10^{-6}	2.33×10^{-5}	5.55×10^{-6}	1.97×10^{-5}	4.40×10^{-6}	1.69×10^{-5}
$p = 0.05$	3.69×10^{-5}	1.18×10^{-4}	2.74×10^{-5}	1.02×10^{-4}	2.22×10^{-5}	8.63×10^{-5}
$p = 0.10$	7.31×10^{-5}	2.42×10^{-4}	5.52×10^{-5}	2.07×10^{-4}	4.50×10^{-5}	1.77×10^{-4}
$p = 0.25$	1.82×10^{-4}	6.58×10^{-4}	1.40×10^{-4}	5.55×10^{-4}	1.16×10^{-4}	4.76×10^{-4}
	$\gamma_0 = 0.01$ (FAR)					
$p = 0.01$	1.23×10^{-5}	3.02×10^{-5}	9.09×10^{-6}	2.54×10^{-5}	6.84×10^{-6}	2.15×10^{-5}
$p = 0.05$	6.15×10^{-5}	1.53×10^{-4}	4.51×10^{-5}	1.31×10^{-4}	3.40×10^{-5}	1.10×10^{-4}
$p = 0.10$	1.22×10^{-4}	3.14×10^{-4}	9.02×10^{-5}	2.66×10^{-4}	6.80×10^{-5}	2.25×10^{-4}
$p = 0.25$	3.03×10^{-4}	8.47×10^{-4}	2.20×10^{-4}	7.08×10^{-4}	1.67×10^{-4}	6.01×10^{-4}
	$\gamma_0 = 0.0027$ (FAR)					
$p = 0.01$	1.52×10^{-5}	3.34×10^{-5}	1.13×10^{-5}	2.80×10^{-5}	8.61×10^{-6}	2.36×10^{-5}
$p = 0.05$	7.59×10^{-5}	1.70×10^{-4}	5.67×10^{-5}	1.44×10^{-4}	4.30×10^{-5}	1.21×10^{-4}
$p = 0.10$	1.52×10^{-4}	3.47×10^{-4}	1.13×10^{-4}	2.92×10^{-4}	8.60×10^{-5}	2.46×10^{-4}
$p = 0.25$	3.79×10^{-4}	9.34×10^{-4}	2.79×10^{-4}	7.78×10^{-4}	2.13×10^{-4}	6.59×10^{-4}
	$\gamma_0 = 0.002$ (FAR)					
$p = 0.01$	1.58×10^{-5}	3.41×10^{-5}	1.18×10^{-5}	2.85×10^{-5}	9.01×10^{-6}	2.41×10^{-5}
$p = 0.05$	7.91×10^{-5}	1.73×10^{-4}	5.92×10^{-5}	1.47×10^{-4}	4.50×10^{-5}	1.23×10^{-4}
$p = 0.10$	1.58×10^{-4}	3.54×10^{-4}	1.19×10^{-4}	2.98×10^{-4}	9.01×10^{-5}	2.51×10^{-4}
$p = 0.25$	3.96×10^{-4}	9.52×10^{-4}	2.92×10^{-4}	7.93×10^{-4}	2.24×10^{-4}	6.71×10^{-4}

Table 10. The LSE bootstrap chart standard errors for the UCL and LCL.

Parameters	$n=4$		$n=5$		$n=6$	
	LCL	UCL	LCL	UCL	LCL	UCL
	$\gamma_0 = 0.1$ (FAR)					
$p = 0.01$	2.29×10^{-6}	2.98×10^{-5}	2.31×10^{-6}	2.37×10^{-5}	2.28×10^{-6}	1.96×10^{-5}
$p = 0.05$	1.18×10^{-5}	1.56×10^{-4}	1.21×10^{-5}	1.22×10^{-4}	1.19×10^{-5}	1.03×10^{-4}
$p = 0.10$	2.48×10^{-5}	3.29×10^{-4}	2.51×10^{-5}	2.64×10^{-4}	2.52×10^{-5}	2.17×10^{-4}
$p = 0.25$	7.17×10^{-5}	1.01×10^{-3}	7.35×10^{-5}	7.94×10^{-4}	7.37×10^{-5}	6.50×10^{-4}
	$\gamma_0 = 0.01$ (FAR)					
$p = 0.01$	1.16×10^{-6}	1.15×10^{-4}	1.33×10^{-6}	7.85×10^{-5}	1.41×10^{-6}	5.86×10^{-5}
$p = 0.05$	5.93×10^{-6}	5.97×10^{-4}	6.85×10^{-6}	3.99×10^{-4}	7.32×10^{-6}	3.08×10^{-4}
$p = 0.10$	1.24×10^{-5}	1.24×10^{-3}	1.41×10^{-5}	8.55×10^{-4}	1.53×10^{-5}	6.37×10^{-4}
$p = 0.25$	3.47×10^{-5}	3.61×10^{-3}	4.06×10^{-5}	2.48×10^{-3}	4.39×10^{-5}	1.84×10^{-3}
	$\gamma_0 = 0.0027$ (FAR)					
$p = 0.01$	8.45×10^{-7}	2.36×10^{-4}	1.04×10^{-6}	1.46×10^{-4}	1.16×10^{-6}	1.03×10^{-4}
$p = 0.05$	4.33×10^{-6}	1.22×10^{-3}	5.30×10^{-6}	7.36×10^{-4}	6.00×10^{-6}	5.40×10^{-4}
$p = 0.10$	8.92×10^{-6}	2.54×10^{-3}	1.09×10^{-5}	1.57×10^{-3}	1.25×10^{-5}	1.12×10^{-3}
$p = 0.25$	2.49×10^{-5}	7.25×10^{-3}	3.09×10^{-5}	4.56×10^{-3}	3.54×10^{-5}	3.17×10^{-3}
	$\gamma_0 = 0.002$ (FAR)					
$p = 0.01$	7.90×10^{-7}	2.69×10^{-4}	9.79×10^{-7}	1.64×10^{-4}	1.11×10^{-6}	1.13×10^{-4}
$p = 0.05$	4.05×10^{-6}	1.39×10^{-3}	5.01×10^{-6}	8.22×10^{-4}	5.80×10^{-6}	6.01×10^{-4}
$p = 0.10$	8.33×10^{-6}	2.90×10^{-3}	1.03×10^{-5}	1.76×10^{-3}	1.20×10^{-5}	1.24×10^{-3}
$p = 0.25$	2.32×10^{-5}	8.23×10^{-3}	2.92×10^{-5}	5.10×10^{-3}	3.40×10^{-5}	3.50×10^{-3}

Table 11. The MMM bootstrap chart standard errors for the UCL and LCL.

Parameters	$n=4$		$n=5$		$n=6$	
	LCL	UCL	LCL	UCL	LCL	UCL
	$\gamma_0 = 0.1$ (FAR)					
$p=0.01$	1.51×10^{-6}	3.12×10^{-5}	1.71×10^{-6}	2.42×10^{-5}	1.92×10^{-6}	2.03×10^{-5}
$p=0.05$	1.03×10^{-5}	2.02×10^{-4}	1.05×10^{-5}	1.47×10^{-4}	1.21×10^{-5}	1.20×10^{-4}
$p=0.10$	2.43×10^{-5}	4.08×10^{-4}	2.41×10^{-5}	2.98×10^{-4}	2.69×10^{-5}	2.35×10^{-4}
$p=0.25$	8.02×10^{-5}	1.15×10^{-3}	7.74×10^{-5}	8.30×10^{-4}	8.39×10^{-5}	6.58×10^{-4}
	$\gamma_0 = 0.01$ (FAR)					
$p=0.01$	0.00+	6.96×10^{-5}	0.00+	4.69×10^{-5}	0.00+	3.58×10^{-5}
$p=0.05$	8.13×10^{-7}	3.70×10^{-4}	1.47×10^{-6}	2.53×10^{-4}	2.50×10^{-6}	2.07×10^{-4}
$p=0.10$	3.97×10^{-6}	7.83×10^{-4}	4.81×10^{-6}	5.46×10^{-4}	6.99×10^{-6}	4.12×10^{-4}
$p=0.25$	1.71×10^{-5}	2.27×10^{-3}	1.89×10^{-5}	1.54×10^{-3}	2.59×10^{-5}	1.14×10^{-3}
	$\gamma_0 = 0.0027$ (FAR)					
$p=0.01$	0.00+	1.10×10^{-4}	0.00+	7.13×10^{-5}	0.00+	5.24×10^{-5}
$p=0.05$	0.00+	5.36×10^{-4}	0.00+	3.72×10^{-6}	0.00+	2.45×10^{-4}
$p=0.10$	1.64×10^{-6}	1.11×10^{-3}	2.35×10^{-6}	7.89×10^{-6}	3.49×10^{-6}	5.40×10^{-4}
$p=0.25$	7.87×10^{-6}	3.22×10^{-3}	1.08×10^{-5}	2.22×10^{-6}	1.36×10^{-5}	1.55×10^{-3}
	$\gamma_0 = 0.002$ (FAR)					
$p=0.01$	0.00+	1.19×10^{-4}	0.00+	7.71×10^{-5}	0.00+	5.63×10^{-5}
$p=0.05$	0.00+	5.76×10^{-4}	0.00+	3.98×10^{-4}	0.00+	2.69×10^{-4}
$p=0.10$	1.25×10^{-6}	1.20×10^{-3}	1.71×10^{-6}	8.37×10^{-4}	3.01×10^{-6}	5.78×10^{-4}
$p=0.25$	6.53×10^{-6}	3.40×10^{-3}	9.26×10^{-6}	2.34×10^{-3}	1.13×10^{-5}	1.63×10^{-3}

Table 12. The ML bootstrap chart standard errors for the UCL and LCL.

Parameters	$n=4$		$n=5$		$n=6$	
	LCL	UCL	LCL	UCL	LCL	UCL
	$\gamma_0 = 0.1$ (FAR)					
$p=0.01$	2.10×10^{-6}	2.70×10^{-5}	2.23×10^{-6}	2.15×10^{-5}	2.39×10^{-6}	1.79×10^{-5}
$p=0.05$	1.09×10^{-5}	1.38×10^{-4}	1.15×10^{-5}	1.10×10^{-4}	1.24×10^{-5}	9.28×10^{-5}
$p=0.10$	2.30×10^{-5}	2.85×10^{-4}	2.44×10^{-5}	2.30×10^{-4}	2.60×10^{-5}	1.92×10^{-4}
$p=0.25$	6.98×10^{-5}	7.98×10^{-4}	7.25×10^{-5}	6.33×10^{-4}	7.49×10^{-5}	5.36×10^{-4}
	$\gamma_0 = 0.01$ (FAR)					
$p=0.01$	1.19×10^{-6}	6.53×10^{-5}	1.17×10^{-6}	4.63×10^{-5}	1.15×10^{-6}	3.50×10^{-5}
$p=0.05$	6.17×10^{-6}	3.33×10^{-4}	6.04×10^{-6}	2.37×10^{-4}	6.07×10^{-6}	1.82×10^{-4}
$p=0.10$	1.29×10^{-5}	4.95×10^{-4}	1.29×10^{-5}	4.95×10^{-4}	1.29×10^{-5}	3.78×10^{-4}
$p=0.25$	3.82×10^{-5}	1.95×10^{-3}	3.90×10^{-5}	1.37×10^{-3}	3.95×10^{-5}	1.06×10^{-3}
	$\gamma_0 = 0.0027$ (FAR)					
$p=0.01$	1.00×10^{-6}	1.04×10^{-4}	1.01×10^{-6}	7.03×10^{-5}	1.01×10^{-6}	5.07×10^{-5}
$p=0.05$	5.18×10^{-6}	5.32×10^{-4}	5.22×10^{-6}	3.57×10^{-4}	5.31×10^{-6}	2.64×10^{-4}
$p=0.10$	1.08×10^{-5}	1.10×10^{-3}	1.11×10^{-5}	7.45×10^{-4}	1.12×10^{-5}	5.46×10^{-4}
$p=0.25$	3.08×10^{-5}	3.12×10^{-3}	3.27×10^{-5}	2.05×10^{-3}	3.36×10^{-5}	1.52×10^{-4}
	$\gamma_0 = 0.002$ (FAR)					
$p=0.01$	9.67×10^{-7}	1.14×10^{-4}	9.95×10^{-7}	7.60×10^{-5}	9.90×10^{-7}	5.46×10^{-5}
$p=0.05$	4.97×10^{-6}	5.80×10^{-4}	5.11×10^{-6}	3.86×10^{-4}	5.21×10^{-6}	2.84×10^{-4}
$p=0.10$	1.03×10^{-5}	1.20×10^{-3}	1.08×10^{-5}	8.04×10^{-4}	1.10×10^{-5}	5.87×10^{-4}
$p=0.25$	2.93×10^{-5}	3.40×10^{-3}	3.18×10^{-5}	2.22×10^{-3}	3.28×10^{-5}	1.63×10^{-3}

Table 13. Out of control ML estimate chart for the generalized Pareto distribution with out of control parameters $\alpha = 5.0$ and $\lambda = 2.5$.

Parameters	$n = 4$		$n = 5$		$n = 6$	
	ARL	SEARL	ARL	SEARL	ARL	SEARL
	$\gamma_0 = 0.1$ (FAR)		$1/\gamma_0 = 10$			
$p = 0.01$	1.3464	0.00706	1.2344	0.00562	1.1528	0.00435
$p = 0.05$	1.3357	0.00689	1.2183	0.00535	1.1515	0.00432
$p = 0.10$	1.3276	0.00686	1.2051	0.00505	1.1248	0.00383
$p = 0.25$	1.2960	0.00653	1.1751	0.00471	1.1023	0.00349
	$\gamma_0 = 0.01$ (FAR)		$1/\gamma_0 = 100$			
$p = 0.01$	3.0699	0.02886	2.3627	0.01966	1.9373	0.01421
$p = 0.05$	3.0619	0.02866	2.3092	0.01887	1.9441	0.01416
$p = 0.10$	3.0602	0.02856	2.2908	0.01862	1.8689	0.01348
$p = 0.25$	2.9571	0.02707	2.1870	0.01750	1.1782	0.01260
	$\gamma_0 = 0.0027$ (FAR)		$1/\gamma_0 = 370.37$			
$p = 0.01$	6.3779	0.07389	4.0018	0.03924	2.9852	0.02778
$p = 0.05$	6.3924	0.07448	3.9766	0.03851	2.9838	0.02724
$p = 0.10$	6.4617	0.07726	3.9563	0.03936	3.69870	0.02615
$p = 0.25$	6.3339	0.07325	3.8642	0.03799	2.7893	0.02583
	$\gamma_0 = 0.002$ (FAR)		$1/\gamma_0 = 500$			
$p = 0.01$	7.9494	0.09725	4.7668	0.04981	3.4133	0.03319
$p = 0.05$	7.9494	0.09697	4.7029	0.04788	4.99500	0.03261
$p = 0.10$	8.2089	0.10630	4.7029	0.04931	3.3173	0.03219
$p = 0.25$	7.9494	0.09476	4.6614	0.05051	4.99500	0.03142

Table 14. Out of control ML estimate charts for the generalized Pareto distribution with out of control parameters $\alpha = 5.0$ and $\lambda = 1.0$

Parameters	$n = 4$		$n = 5$		$n = 6$	
	ARL	SEARL	ARL	SEARL	ARL	SEARL
	$\gamma_0 = 0.1$ (FAR)		$1/\gamma_0 = 10$			
$p = 0.01$	4.5318	0.0438	4.0891	0.0391	3.8218	0.0349
$p = 0.05$	4.5271	0.0439	4.0515	0.0386	3.6853	0.0338
$p = 0.10$	4.4212	0.0429	3.9901	0.0375	3.6906	0.0344
$p = 0.25$	4.2499	0.0413	3.6662	0.0352	3.3230	0.0307
	$\gamma_0 = 0.01$ (FAR)		$1/\gamma_0 = 100$			
$p = 0.01$	25.8143	0.3380	20.2286	0.2511	16.6361	0.1876
$p = 0.05$	25.5503	0.3208	20.1428	0.2450	16.5277	0.1892
$p = 0.10$	25.5291	0.3254	19.6476	0.2364	16.0774	0.1877
$p = 0.25$	25.1304	0.3175	19.1412	0.2393	15.3536	0.1731
	$\gamma_0 = 0.0027$ (FAR)		$1/\gamma_0 = 370.37$			
$p = 0.01$	82.2897	1.1592	58.3665	0.8167	43.1321	0.5541
$p = 0.05$	80.8592	1.0892	57.2253	0.7818	43.2578	0.5656
$p = 0.10$	82.1383	1.1715	57.2081	0.7799	43.2304	0.5705
$p = 0.25$	81.1916	1.1376	56.9895	0.8237	41.4291	0.5506
	$\gamma_0 = 0.002$ (FAR)		$1/\gamma_0 = 500$			
$p = 0.01$	114.3089	1.7078	78.5059	1.1334	57.2333	0.8035
$p = 0.05$	112.5334	1.6530	78.3478	1.1540	57.5751	0.7900
$p = 0.10$	115.9670	1.7793	78.3478	1.1379	57.8656	0.8186
$p = 0.25$	112.5334	1.6395	78.3928	1.2019	55.6368	0.7737

Table 15. Out of control ML estimate charts for the generalized Pareto distribution with out of control parameters $\alpha = 2.5$ and $\lambda = 2.0$

Parameters	$n = 4$		$n = 5$		$n = 6$	
	ARL	SEARL	ARL	SEARL	ARL	SEARL
	$\gamma_0 = 0.1$ (FAR)		$1/\gamma_0 = 10$			
$p = 0.01$	4.8353	0.0463	4.4966	0.0442	4.1259	0.0390
$p = 0.05$	4.8245	0.0471	4.3706	0.0429	4.0561	0.0376
$p = 0.10$	4.7334	0.0447	4.3496	0.0420	3.9753	0.0373
$p = 0.25$	4.6432	1.7211	4.1408	0.0401	3.7211	0.0352
	$\gamma_0 = 0.01$ (FAR)		$1/\gamma_0 = 100$			
$p = 0.01$	28.5761	0.3498	23.0087	0.2683	19.8312	0.2316
$p = 0.05$	28.6142	0.3493	22.8181	0.2667	19.6298	0.2276
$p = 0.10$	28.1088	0.3444	23.4356	0.2844	19.3925	0.2204
$p = 0.25$	28.5066	0.3539	22.4157	0.2812	18.4769	0.2127
	$\gamma_0 = 0.0027$ (FAR)		$1/\gamma_0 = 370.37$			
$p = 0.01$	88.7289	0.3498	67.6586	0.9705	52.6793	0.6839
$p = 0.05$	89.0722	1.2382	66.6053	0.8885	52.3929	0.6743
$p = 0.10$	88.4970	1.2085	66.2546	0.8827	50.8437	0.6362
$p = 0.25$	88.8254	1.2125	65.7238	0.9371	50.2462	0.6486
	$\gamma_0 = 0.002$ (FAR)		$1/\gamma_0 = 500$			
$p = 0.01$	122.4647	1.9951	93.4767	1.4172	69.6476	0.9444
$p = 0.05$	122.9314	1.7426	90.2651	1.2825	69.6476	0.9432
$p = 0.10$	120.3574	1.6882	90.2651	1.2781	68.2267	0.8971
$p = 0.25$	122.3561	1.7211	89.0618	1.2991	67.4059	0.9072

4.3 Illustrative example

Assume certain machine parts have failure times in terms of years that have a generalized Pareto distribution with $\alpha = 2.5$ and $\lambda = 1.0$. Since no real-world data could be obtained during this study, an R program was created to generate twenty subgroups with six machine part lifetimes a piece. Since λ is a rate parameter, without the loss of generality λ can be selected as one with a reasonable measurement unit in lifetime measure. α is the shape parameter that should not be too large or too small. These subgroups were made independently from the in-control generalized Pareto distribution with $\alpha = 2.5$ and $\lambda = 1.0$. These twenty subgroups are reported in Table 16. The designer of the parts is concerned about the tenth percentile of the lifetime of his parts, $Q(0.10; \alpha_0, \lambda_0)$. After the first twenty subgroups, assume that the process was shifted to out of control where $\alpha_1 = 5.0$ and $\lambda_1 = 2.5$ and another twenty subgroups were generated with six machine part lifetimes a piece. These twenty subgroups are displayed in Table 17. The ML bootstrap chart was developed based on the twenty in-control subgroups in Table 16 where the FAR=0.0027 and B=10,000. The control limits were are LCL $= 0.0129$ and UCL $= 0.1646$. The center line is CL $= 0.06007$. Figure 1 (top) shows the control chart for the in-control percentiles and Figure 1 (bottom) shows the same control chart for monitoring the out of control tenth percentile. In Figure 1 (top) all of the tenth percentiles calculated based on twenty subgroups, respectively, are within the control limits and spread around the CL. In Figure 1 (bottom), notice that the first tenth percentile signals an out of control process. While not all of the tenth percentiles are outside of the control chart limits in this figure, they are grouped rather tightly and are all well below the CL. Thus, the ML bootstrap chart is successful in indicating that a process is out of control.

Table 16. Twenty subgroups of machine part lifetimes generated from the generalized Pareto distribution with $\alpha_0 = 2.5$ and $\lambda_0 = 1.0$.

Subgroup number	Lifetime observations					
1	0.5819	2.3860	0.1465	0.2941	0.3153	0.1461
2	1.1770	0.0462	0.1149	0.7403	0.2062	1.8740
3	0.0467	0.1225	0.0617	0.2224	0.9333	0.4332
4	5.2410	0.8200	0.4782	0.5241	0.0439	1.3510
5	0.6481	0.0607	0.1688	0.1478	0.8709	0.2992
6	0.9731	0.0956	0.0493	1.6600	0.3200	0.3037
7	0.1695	0.0998	1.2960	0.0525	0.0936	1.3860
8	0.2028	0.0197	0.8517	0.7443	0.6432	0.1275
9	0.0614	0.1126	0.1307	0.0167	0.5010	1.2790
10	0.6115	0.5098	1.0260	0.9001	0.2065	0.0695
11	1.7350	2.1580	1.1040	0.2383	0.3030	0.1099
12	0.0758	0.4705	0.0119	0.1444	0.0568	0.8328
13	1.2990	0.6935	0.2923	0.7409	0.4427	0.8387
14	0.7045	0.1364	3.5530	0.0713	0.1115	0.5185
15	0.2578	0.4141	0.2453	1.6400	0.2592	0.3155
16	1.3130	0.0240	1.1280	0.0591	0.1310	0.0676
17	0.5947	0.0189	0.4675	0.0356	1.4630	0.0643
18	0.2588	0.1155	0.4547	1.2500	0.7298	0.1451
19	0.1267	1.2390	0.0508	0.2061	0.2859	1.2500
20	0.3479	0.0243	0.2715	0.0724	0.0877	1.3420

Table 17. Twenty subgroups of machine part lifetimes generated from the generalized Pareto distribution with $\alpha_1 = 5.0$ and $\lambda_1 = 2.5$.

Subgroup number	Lifetime observations					
1	0.0342	0.0787	0.2626	0.0460	0.0135	0.0040
2	0.2904	0.0627	0.0198	0.0252	0.0727	0.1682
3	0.0049	0.2299	0.0117	0.0375	0.0613	0.0088
4	0.0402	0.2255	0.0342	0.0958	0.1299	0.0905
5	0.0450	0.0422	0.2306	0.1699	0.0893	0.0174
6	0.0170	0.4364	0.2594	0.0518	0.2007	0.1366
7	0.0547	0.0112	0.0004	0.0023	0.0761	0.0094
8	0.0613	0.0013	0.7946	0.0365	0.1964	0.1364
9	0.0018	0.1491	0.0472	0.1392	0.1302	0.0829
10	0.0345	0.0032	0.0227	0.0420	0.0975	0.1786
11	0.0199	0.0141	0.0103	0.0709	0.0095	0.2356
12	0.0220	0.7481	0.0402	0.1396	0.0129	0.0989
13	0.0166	0.0034	0.0148	0.1722	0.2251	0.0620
14	0.0038	0.1211	0.2050	0.0061	0.2040	0.0528
15	0.0093	0.0105	0.0855	0.0156	0.1116	0.0153
16	0.0045	0.0648	0.2079	0.0912	0.0727	0.0258
17	0.1148	0.1332	0.1420	0.2850	0.0859	0.0154
18	0.1863	0.1126	0.2125	0.0102	0.0781	0.1808
19	0.0356	0.2005	0.0333	0.2909	0.1731	0.6069
20	0.0719	0.0411	0.1251	0.0564	0.4224	0.1394

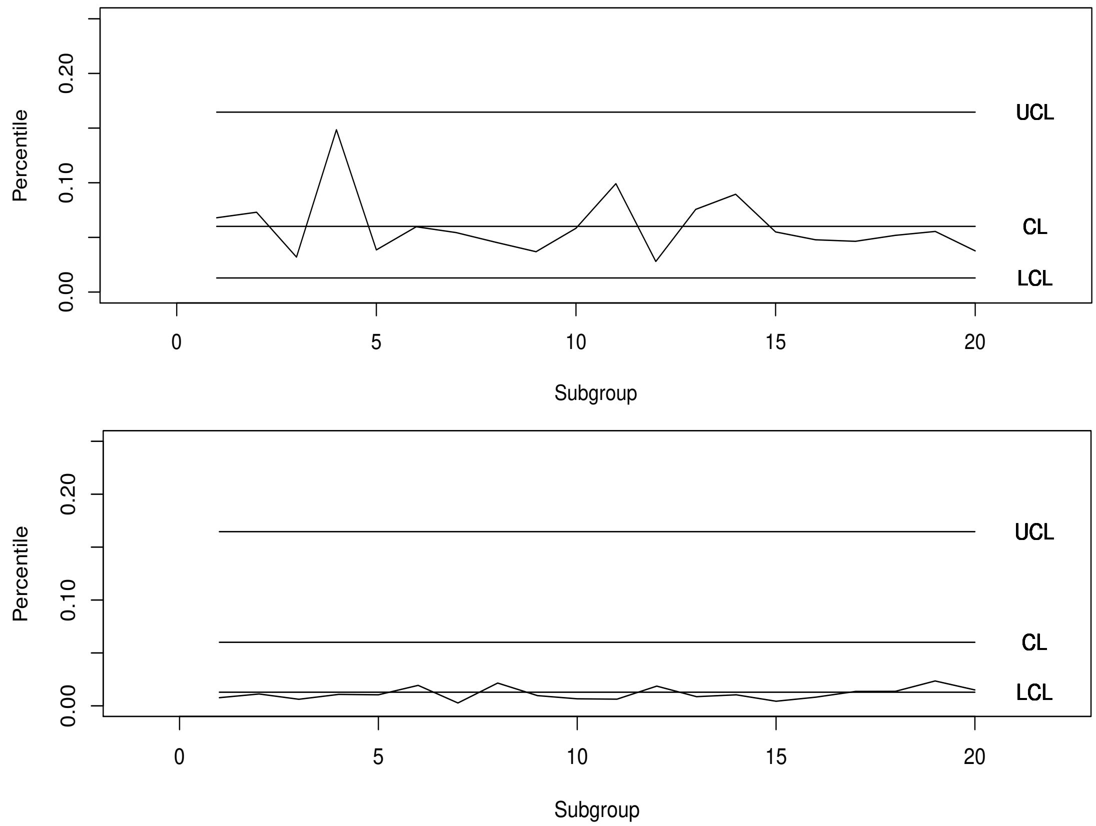

Figure 1. In control subgroups (top) and out control subgroups with FAR=0.0027.

5. Conclusions

In order to examine the Pareto percentiles, the Shewhart-type chart and three parametric bootstrap charts were constructed. As a result of the Monte Carlo simulation, it was discovered that the Shewhart-type chart was inadequate in providing appropriate control limits. Two of the parametric bootstrap charts were also shown to be unsuitable for providing appropriate control limits. The least squared error control chart consistently overestimated the nominal average run length and the modified moment method chart was inconsistent. However the maximum likelihood chart was shown to be acceptable choice for monitoring Pareto percentiles. As it also promptly detects an out of control process, as shown in the simulation and illustrative example, it is recommended for practical use.

It should be mentioned that the conclusions from the current research may not be applied to any other case with shape parameter, α, or rate parameter, λ, too far from the current values for generalized Pareto distribution. However, the current simulation procedures provides a guideline to run simulation study for different α and λ to make a selection of control chart method. When a real word application data are given, it is suggested to use Kolmogorov-Smirnov test with Akaike and Bayesian information criteria to select probability model. Then, the practitioners can follow the current research procedure to decide the control chart method after the lifetime distribution has been decided. Further research for multivariate control charts under non-normality can be explored (Marchant et al., 2019, 2018).

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References

Aykroyd, R.G., Leiva, V., and Ruggeri, F., 2019. Recent developments of control charts, identification of big data sources and future trends of current research. Technological Forecasting and Social Change, 144, 221–232.

Chen, H., Cheng, W., Zhao, J., and Zhao, X., 2017. Parameter estimation for generalized Pareto distributions by generalized probability weighted moment-equations. Communications in Statistics: Simulation and Computation, 46, 7761–7776.

Chiang, J., Jiang, N., Brown, T., Tsai, T., and Lio, Y., 2017. Control charts for generalized exponential distribution percentiles. Communication in Statistics: Simulation and Computation, 46, 7827–7843.

Efron, B., Tibshirani, R., 1993. An Introduction to the Bootstrap. Chapman and Hall, New York.

Gunter, B., 1992. Bootstrapping: How to make something from almost nothing and get statistically valid answers, part III. Quality Progress, 25, 119–122.

Hosking, J. and Wallis, J., 1987. Parameter and quantile estimation for the generalized Pareto distribution. Technometrics, 29, 339–349.

Hüsler, J., Li, D., and Raschke, M., 2011. Estimation for the generalized Pareto distribution using maximum likelihood and goodness of fit. Communications in Statistics: Theory and Methods, 40, 2500–2510.

Ihaka, R. and Gentleman, R., 1996. A language for data analysis and graphics. Journal of Computational and Graphical Statistics, 5, 299–314.

Lio, Y., Park, C., 2008. A bootstrap control chart for Birnbaum-Saunders percentiles. Quality and Reliability Engineering International, 24, 585–600.

Lio, Y. and Park, C., 2010. A bootstrap control chart for inverse Gaussian percentiles. Journal of Statistical Computation and Simulation, 80, 287–299.

Lio, Y., Tsai, T., Aslam, M., and Jiang, N., 2014. Control charts for monitoring Burr type-X percentiles. Communication in Statistics: Simulation and Computation, 43, 761–776.

Marchant, C., Leiva, V., Christakos, G., and Cavieres, M.F., 2019. Monitoring urban environmental pollution by bivariate control charts: new methodology and case study in Santiago, Chile. Environmetrics, 30, e2551.

Marchant, C., Leiva, V., Cysneiros, F.J.A., and Liu, S., 2018. Robust multivariate control charts based on Birnbaum-Saunders distributions. Journal of Statistical Computation and Simulation, 88, 182–202.

Nichols, M. and Padgett, W., 2005. A bootstrap control chart for Weibull percentiles. Quality and Reliability Engineering International, 22, 141–151.

Pickands, J., 1975. Statistical inference using extreme order statistics. The Annals of Statistics, 29, 119–131.

Rezac, J., Lio, Y., and Jiang, N., 2015. Burr type-XII percentile control charts. Chilean Journal of Statistics, 6(1), 67–87.

Salmasi, M. and Yari, G. 2017. On generalized order statistics from generalized Pareto distribution. Communications in Statistics: Simulation and Computation, 46, 5682–5697.

Wang, B., 2008. Statistical inference for the Burr type-XII distribution. ACTA Mathematica Scientia, 28, 1103–1108.

Young, G., 1994. Bootstrap: More than a stab in the dark. Statistical Science, 9, 382–415.

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