CHILEAN
JOURNAL OF
STATISTICS

Edited by Víctor Leiva

Volume 10 Number 1
April 2019
ISSN: 0718-7912 (print)
ISSN: 0718-7920 (online)

Published by the
Chilean Statistical Society
SOCHE
SOCIEDAD CHILENA DE ESTADÍSTICA

Chilean Journal of Statistics
Vol. 10, No. 1, April 2019, 77–94

Distribution Theory
Research Paper

The Harris extended Lindley distribution for modeling hydrological data

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(Received: 17 April 2018 · Accepted in final form: 14 February 2019)

Abstract

We introduce a three-parameter extension of the Lindley distribution, which has as sub-models the Lindley and Marshall-Olkin Lindley distributions. The proposed model turns out to be quite flexible: its probability density function can be decreasing or unimodal and its associated hazard rate may be increasing, decreasing, unimodal or bathtub-shaped. Since this new distribution has a survival function and a hazard rate that can be expressed in closed form, it can readily be simulated and used to analyze censored data. Computable expressions are obtained for certain statistical functions such as its quantile function, ordinary and incomplete moments, moment generating function, order statistics and reliability function. The maximum likelihood method is utilized to obtain estimates of the model parameters and a simulation study is carried out to assess the performance of the corresponding maximum likelihood estimators. Two illustrative examples involving hydrological data sets are presented.

Keywords: Data modeling · Extended distributions · Hazard rate · Maximum likelihood estimation · Monte Carlo simulations · Precipitation data.

Mathematics Subject Classification: Primary 60E05 · Secondary 62E10 · 62N05

1. Introduction

Lindley (1958) introduced a one-parameter distribution in the context of fiducial and Bayesian statistics, which is obtained as a mixture of exponential(λ) and gamma(2, λ) probability density functions (PDFs), as defined in Equation (2). Aly and Benkherouf (2011) recently proposed a convenient method for adding two parameters to a baseline distribution, which gives rise to what is referred to as the Harris extended (HE) family of distributions. This family includes the baseline distribution itself as a basic exemplar and provides more flexibility for modeling various types of data. This novel approach is based on the probability generating function of a discrete distribution introduced by Harris (1948). In this paper, we define a three-parameter generalization of the Lindley distribution by applying to it the HE generator, the resulting model being named the Harris extended Lindley (HEL) distribution. This distribution is in fact an extension of the Marshall-Olkin

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ISSN: 0718-7912 (print)/ISSN 0718-7920 (online)

http://www.soche.cl/chjs

extended Lindley (MOL) distribution that was proposed by Ghitany et al. (2012), and its additional shape parameter α ought to provide an improved fit related to the MOL distribution. This extra parameter helps in controlling the shape of the HE PDF and enables us to model heavy-tailed distributions which are fairly common in hydrology; see, e.g., Li et al. (2013) and Ashkar and El Adlouni (2014). Moreover, the new distribution has an interesting physical interpretation when α is a positive integer and $0 < \theta < 1$: it is indeed the distribution of the time until failure of a device composed of N serial components having constant failure rate, where N is a random variable which arises from a branching process such as that described in Harris (1948). This distribution can be utilized for modeling purposes in research fields such as hydrology, engineering, insurance, biology and epidemiology wherein skewed positive data are frequently encountered.

One of the most crucial aspects of hydrological data analysis consists in achieving a close fit to the experimental data by employing proper statistical models. The Gumbel, Weibull, gamma, generalized logistic as well as other well-known distributions have been extensively utilized for modeling hydrological observations such as rainfall, flood, precipitation and stream flow data; see, e.g., Zelenhasic (1970), Chadwick et al. (2004), Heo and Boes (2011), Bhunya et al. (2012) and Kang et al. (2015). Yet, there exists a need for developing more flexible statistical models that would be applicable to data sets related to hydrological structures and phenomena or water resource planning and management, and the proposed three-parameter generalization of the Lindley distribution fits the purpose.

Although little attention has been paid to the Lindley distribution, there has recently been a surge of interest in this model, generalizations thereof and related applications. Nadarajah et al. (2007) introduced the exponentiated Lindley distribution as an alternative to the gamma, log-normal, Weibull and exponentiated exponential distributions; see also Cordeiro et al. (2016). Several properties of the Lindley distribution have been studied by Ghitany et al. (2008) who have shown that, for instance, it can provide a better fit than the exponential distribution. Ghitany et al. (2011) studied another two parameter extension of Lindley distribution and called it the weighted Lindley distribution. By making use of the Marshall-Olkin method, Ghitany et al. (2012) introduced and studied another extension of the Lindley model called the Marshall-Olkin extended Lindley (MOL) distribution. Ghitany et al. (2013) introduced a two-parameter power Lindley distribution and discussed its properties. A three-parameter generalization of the Lindley model was introduced by Mervoci and Sharma (2014). This extension, referred to as the beta Lindley (BL) distribution, is generated from the logit of a beta random variable. Ghitany et al. (2015) considered the problem of estimating the stress-strength parameter of the power Lindley distribution. Mazucheli et al. (2016) developed some statistical for testing hypotheses on the parameters of the weighted Lindley distribution. Alizadeh et al. (2017) introduced another extension of the power Lindley distribution.

The objective of this work is to derive the HEL distribution focusing on its probabilistic and statistics aspects, as well as applications in hydrology.

The remainder of the paper is organized as follows. We define the new distribution in Section 2. In Section 3, we provide computable expressions for some of its statistical functions such as its quantile function (QF), ordinary and incomplete moments, mean deviations, moment generating function (MGF) and order statistics. In Section 4, the model parameters are estimated by making use of the maximum likelihood (ML) method and a simulation study is carried out. In Section 5, we illustrate the usefulness of the proposed distribution by modeling two hydrological data sets. Finally, Section 6 offers some concluding remarks.

2. The HEL Distribution

In this section, we provide probabilistic aspects of the HEL distribution. The survival function (SF) and PDF of the distribution introduced by Lindley (1958) are respectively given by

$$\bar{G}_{\rm L}(x) = \left(\frac{1+\lambda+\lambda\,x}{1+\lambda}\right) {\rm e}^{-\lambda x}, \quad x > 0, \tag{1}$$

and

$$g_{\rm L}(x) = \frac{\lambda^2}{\lambda+1}(1+x)\,{\rm e}^{-\lambda x}, \quad x > 0, \tag{2}$$

where the parameter λ is assumed to be positive. We now describe a technique whereby the so-called Harris extended family of distributions can be generated and apply it to the Lindley distribution. The resulting distribution is referred to as the Harris extended Lindley (HEL) distribution. Let $G(x) = G(x;\xi)$ be a baseline cumulative distribution function (CDF) and

$$\bar{G}(x) = \bar{G}(x;\xi) = 1 - G(x;\xi)$$

be the corresponding SF of a lifetime random variable W, where $\xi = (\xi_1, \ldots, \xi_q)$ is a parameter vector of dimension q. Furthermore, let $g(x) = g(x;\xi)$ be the PDF of W. The SF of the HE family is then defined by

$$\bar{F}_{\rm HE}(x) = \frac{\theta^{1/\alpha}\,\bar{G}(x)}{\left[1-\bar{\theta}\bar{G}(x)^{\alpha}\right]^{1/\alpha}}, \quad x > 0, \tag{3}$$

where $\bar{\theta} = 1-\theta$, the parameters $\theta > 0$ and $\alpha > 0$ being additional shape parameters that allow for greater flexibility. Thereupon, the HE PDF has the form

$$f_{\rm HE}(x) = \frac{\theta^{1/\alpha}\,g(x)}{\left[1-\bar{\theta}\bar{G}(x)^{\alpha}\right]^{1+1/\alpha}}, \quad x > 0.$$

Aly and Benkherouf (2011) pointed out that when $\alpha > 0$ is a positive integer, the HE family can be looked upon as resulting from examining a simple discrete branching process where a particle either splits into $(\alpha+1)$ identical branches or remains the same during a short interval. Clearly, Equation (3) constitutes a flexible generator for obtaining new parametric distributions from existing ones. For $\theta = 1$, $\bar{F}(x) = \bar{G}(x)$ and $\bar{G}(x)$ is thus a basic exemplar of the distribution. Additionally, the Marshall and Olkin (1997) extended (MOE) family arises from Equation (3) by letting $\alpha = 1$. Accordingly, the HE family can be viewed as a generalization of the MOE family.

The SF of the HEL distribution is defined as

$$\bar{F}(x) = \frac{\theta^{1/\alpha}\bar{G}_{\rm L}(x)}{\left[1-\bar{\theta}\bar{G}_{\rm L}(x)^{\alpha}\right]^{1/\alpha}}, \quad x > 0, \tag{4}$$

for $\alpha > 0$, $\theta > 0$, $\lambda > 0$, where $\bar{G}_{\rm L}(x)$ is given in Equation (1), with its PDF corresponding to Equation (4) being

$$f(x) = \frac{\theta^{1/\alpha}\,\lambda^2\,(1+x)\,\mathrm{e}^{-\lambda x}}{(1+\lambda)\left[1-\bar{\theta}\,\bar{G}_{\mathrm{L}}(x)^{\alpha}\right]^{1+1/\alpha}}, \quad x>0. \tag{5}$$

Henceforth, a random variable X having the PDF specified in Equation (5) is denoted by $X \sim \mathrm{HEL}(\theta,\alpha,\lambda)$. This three-parameter PDF has two shape parameters and one scale parameter, and it can be either decreasing or unimodal. The two main special cases of the HEL model are: (i) the MOL distribution in which case $\alpha = 1$; (ii) the Lindley distribution which is obtained by letting $\alpha = \theta = 1$. The hazard rate (HR) associated with HEL model is given by

$$h(x) = \frac{\lambda^2\,(1+x)}{(\lambda+1+\lambda\,x)}\left[1-\bar{\theta}\,\bar{G}_{\mathrm{L}}(x)^{\alpha}\right]^{-1}, \quad x>0.$$

This HR can assume the four principal shapes associated with increasing, decreasing, bathtub-shaped or upside-down bathtub-shaped HRs. The HEL model is thus most appropriate to analyze a variety of hydrological and lifetime data sets. We note that there appears to be very few three-parameter distributions in the literature whose HR can take on the four main shapes of an HR. Moreover, the SF and HR of the HEL distribution have closed-form representations. Accordingly, this model can readily be utilized to analyze censored data sets. As well, simulating it is straightforward.

Figures 1 and 2 display some plots of the PDF and HR of the HEL distribution for certain parameter values. Figure 1 indicates that the HEL PDF can be right-skewed and reversed-J shaped. Figure 2 reveals that the HEL HR can be increasing (IFR), decreasing (DFR), upside-down bathtub (UBT) or bathtub-shaped (BT).

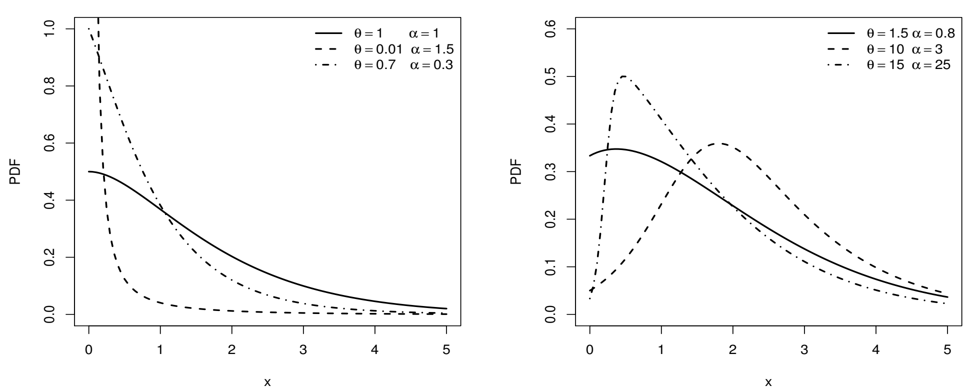


Figure 1. Plots of the HEL PDF for certain parameter values.

Given the functional form of the HEL PDF denoted by $f(x)$, a general representation of the mode that would be expressible in terms of the parameters of the distribution does not appear to be tractable. However, for a specific set of parameters, the command `NSolve[f'[x]==0,x,Reals]` in `Mathematica` can readily be utilized to determine the mode. If the solution happens to be greater than zero, then the PDF has a mode at that point; otherwise, it is strictly decreasing on the positive half-line. The extremum of the HR can be similarly obtained whenever it exists.

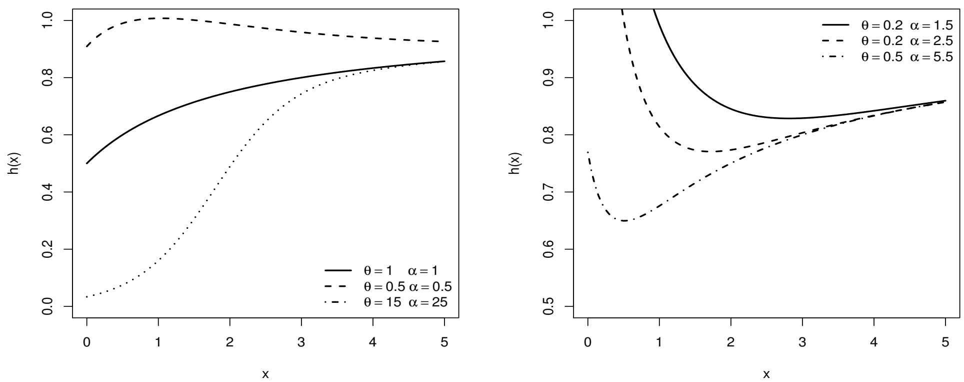


Figure 2. Plots of the HEL HR for certain parameter values.

3. Statistical Functions of the HEL Distribution

In this section, we provide computable representations of certain statistical functions of the HEL distribution. More specifically, we focus, in order, on the quantile function, some useful expansions, the moments, including the incomplete ones, the moment generating function and the order statistics. The derived expressions can be easily evaluated by most symbolic computation software packages such as `Maple`, `Mathematica` and `Matlab`. These platforms can process analytic expressions of great complexity. Whenever available, an explicit representation of a statistical function is preferable to its determination by numerical integration.

The QF of a distribution has numerous uses in both statistical theory and applications. In the case of the HEL distribution, its QF is obtained by inverting the HEL CDF and is given by

$$Q(u) = -1 - \frac{1}{\lambda} - \frac{1}{\lambda} W\left[-(1+\lambda)\frac{1-\tau}{\mathrm{e}^{1+\lambda}}\right], \quad 0 < u < 1, \tag{6}$$

where $\tau = 1-(1-u)\left[\theta + \bar{\theta}(1-u)^{\alpha}\right]^{-1/\alpha}$ and $W(x)$ is the negative branch of the Lambert W function, see Corless et al. (1996) and Jodrá (2010) for details on its properties. The Lambert function cannot be expressed in terms of elementary functions. However, it is analytically differentiable and integrable and its principal branch satisfies $x = W(x\,\mathrm{e}^{x}), x \geq -1$. Furthermore, whenever $|x| \leq \mathrm{e}^{-1}$, $W(x) = \sum_{n=1}^{\infty}(-n)^{n-1}x^{n}/n$. Clearly, if U has a uniform distribution in the interval $(0,1)$, then $X = Q(U)$ has the PDF specified in Equation (5). The Lambert W function is implemented within various scientific libraries, as for example, in the `R` software (by the `lamW` package), `Mathematica` (by the `ProductLog` function), `Matlab` (by the `lambertw` function) and `Maple` (by the `LambertW` function), thus allowing for efficient evaluation of the QF of the HEL distribution.

Some useful expansions are now provided. Let $g_a(x) = a\,g(x)\bar{G}(x)^{a-1}$ be the Lehmann type-II-G (LII-G) PDF with power parameter $a > 0$. We demonstrate that the HEL PDF can be expressed as a linear combination of LII-Lindley (LIIL) PDFs. First, for $0 < \theta < 1$, we consider the negative binomial series

$$(1-z)^{-p} = \sum_{i=0}^{\infty} \frac{\Gamma(p+i)}{\Gamma(p)\,i!}\,z^{i},$$

which holds for $|z| < 1$ and any real number $p > 0$, where $\Gamma(a) = \int_0^{\infty} z^{a-1}\mathrm{e}^{-z}\mathrm{d}z$ is the complete gamma function. Using this power series in Equation (5), we have

$$f(x) = \theta^{1/\alpha}\, g_{\rm L}(x) \sum_{j=0}^{\infty} \bar{\theta}^j\, \frac{\Gamma(\alpha^{-1}+1+j)}{\Gamma(\alpha^{-1}+1)j!}\, \bar{G}_{\rm L}(x)^{j\,\alpha},$$

where $\bar{G}_{\rm L}(x)$ and $g_{\rm L}(x)$ are the SF and PDF of the Lindley distribution as provided by Equations (1) and (2). Note that for $\theta > 1$, we can write

$$f(x) = \theta^{-1} g_{\rm L}(x) \sum_{j=0}^{\infty}\sum_{\ell=j}^{\infty} (-1)^j \left(\frac{\theta-1}{\theta}\right)^{\ell} \binom{\ell}{j} \frac{\Gamma(\alpha^{-1}+1+\ell)}{\Gamma(\alpha^{-1}+1)\ell!}\, \bar{G}_{\rm L}(x)^{j\,\alpha}.$$

On combining the last two expressions for $f(x)$ in a single one, we have

$$f(x) = \sum_{j=0}^{\infty} w_j\, h_{j\,\alpha+1}(x), \tag{7}$$

where $h_{j\,\alpha+1}(x) = (j\,\alpha+1)\, g_{\rm L}(x)\, \bar{G}_{\rm L}(x)^{j\,\alpha}$ is the LIIL PDF with power parameter $j\,\alpha+1$ and

$$w_j = w_j(\alpha,\theta) = \begin{cases} \frac{\theta^{1/\alpha}\,\bar{\theta}^j\,\Gamma(\alpha^{-1}+1+j)}{(j\,\alpha+1)\,\Gamma(\alpha^{-1}+1)j!},\ 0<\theta<1 \\ \frac{(-1)^j\,\theta^{-1}}{(j\,\alpha+1)} \sum_{\ell=j}^{\infty} (\frac{\theta-1}{\theta})^{\ell} \binom{\ell}{j} \frac{\Gamma(\alpha^{-1}+1+\ell)}{\Gamma(\alpha^{-1}+1)\ell!},\ \theta>1. \end{cases}$$

Equation (7) reveals that the HEL PDF (for any $\theta > 0$) can indeed be expressed as a linear combination of LIIL PDFs. It can also be shown that the HEL PDF can be expressed as a linear combination of gamma PDFs. Given Equations (1) and (2), it follows from the representation of Equation (7) that

$$f(x) = \sum_{j=0}^{\infty} w_j\,(j\,\alpha+1)\left(\frac{\lambda^2}{\lambda+1}\right)(1+x)\left(1+\frac{\lambda\,x}{1+\lambda}\right)^{j\,\alpha} {\rm e}^{-(j\,\alpha+1)\lambda x}.$$

On expanding $[1+\lambda\,x/(1+\lambda)]^{j\,\alpha}$ and using the Taylor series $z^{\beta} = \sum_{k=0}^{\infty}(\beta)_k\,(z-1)^k/k!$, where $(\beta)_k = \beta(\beta-1)\cdots(\beta-k+1)$ is the falling factorial, after some algebra, we obtain

$$f(x) = \sum_{i,j=0}^{\infty} v_{i,j} x^i\,(1+x)\,{\rm e}^{-(j\,\alpha+1)\lambda x}, \tag{8}$$

where $v_{i,j} = (j\,\alpha+1)\,w_j\,\left[\lambda^{2+i}/(\lambda+1)^{i+1}\right]\,(j\,\alpha)_i/i!$ for $i,j = 0,1,2,\ldots$.

Letting $\pi(x;\alpha,\beta) = \beta^{\alpha}\,x^{\alpha-1}\,{\rm e}^{-\beta x}/\Gamma(\alpha)$ be the gamma PDF with shape parameter $\alpha > 0$ and rate parameter $\beta > 0$, we can then rewrite Equation (8) as

$$f(x) = \sum_{i,j=0}^{\infty}\left[v_{i,j}^{(1)}\,\pi\,(x; i+1, (j\,\alpha+1)\lambda) + v_{i,j}^{(2)}\,\pi\,(x; i+2, (j\,\alpha+1)\lambda)\right], \tag{9}$$

where $v_{i,j}^{(1)} = i!\,v_{i,j}/[(j\,\alpha+1)\lambda]^{i+1}$ and $v_{i,j}^{(2)} = (i+1)!\,v_{i,j}/[(j\,\alpha+1)\lambda]^{i+2}$.

Equation (9) indicates that the HEL PDF can also be expressed as a linear combination of gamma PDFs. Thus, this representation can be used to obtain explicit expressions for the ordinary and incomplete moments and the MGF of the HEL distribution from the corresponding quantities associated with the gamma distribution. Equations (7) and (9) constitute the main results of this section.

Certain of the main characteristics of a distribution such as tendency, dispersion, skewness and kurtosis can be investigated via its moments. We now establish that the ordinary moments of the HEL distribution can be obtained as infinite power series. It follows from Equation (7) that

$$\mu'_r = \mathrm{E}(X^r) = \frac{\lambda^2}{1+\lambda}\sum_{j=0}^{\infty} w_j \int_0^{\infty} x^r (1+x)\left(1+\frac{\lambda\, x}{1+\lambda}\right)^{j\,\alpha} \mathrm{e}^{-\lambda\,(j\,\alpha+1)x}\mathrm{d}x,$$

or equivalently

$$\mu'_r = \frac{\lambda^2}{1+\lambda}\sum_{j=0}^{\infty} w_j \int_0^{\infty} x^r (1+x)\sum_{i=0}^{\infty}\left(\frac{\lambda}{1+\lambda}\right)^{i} x^i\,\frac{(j\,\alpha)_i}{i!}\,\mathrm{e}^{-\lambda\,(j\,\alpha+1)\,x}\mathrm{d}x.$$

After some algebra, we obtain

$$\mu'_r = \frac{\lambda^2}{1+\lambda}\sum_{i,j=0}^{\infty} p_{i,j}\,\frac{\Gamma(r+i+1)}{[\lambda(j\,\alpha+1)]^{r+i+1}}\left(1+\frac{r+i+1}{\lambda(j\,\alpha+1)}\right), \tag{10}$$

where $p_{i,j} = w_j\,[(j\,\alpha)_i/i!]\,(\lambda/(1+\lambda))^i$.

Table 1 includes numerical values for the first four ordinary moments of the HEL distribution as evaluated from Equation (10) by truncating the series to 100 terms and computed by numerical integration for some parameter values. We note that the numerical values obtained from both approaches are consistently in close agreement.

Table 1. Ordinary moments of the HEL distribution for certain parameter values with $\lambda = 10$.

	$\alpha = 0.5$		$\alpha = 1.5$	
μ'_r	Numerical	Equation (10)	Numerical	Equation (10)
		$\theta = 0.5$		
μ'_1	0.0670906	0.0670905	0.0833919	0.08156687
μ'_2	0.0105268	0.01052653	0.0158697	0.01586975
μ'_3	0.00276376	0.002763106	0.00492889	0.004928885
μ'_4	0.0010382	0.001036676	0.0020813	0.002081299
		$\theta = 1.5$		
μ'_1	0.141446	0.1414455	0.127601	0.1276013
μ'_2	0.0364545	0.0364543	0.0295221	0.02952214
μ'_3	0.0132554	0.01325516	0.0098071	0.009807097
μ'_4	0.00616152	0.006160951	0.00425269	0.004252694

The rth incomplete moment of X is given by $m_r(y) = \int_0^y x^r\, f(x)\mathrm{d}x$. On making use of Equation (7) and proceeding as in the case of ordinary moments, we obtain

$$m_r(y) = \frac{\lambda^2}{1+\lambda}\sum_{j,i=0}^{\infty} w_j \left(\frac{\lambda}{1+\lambda}\right)^{i}\frac{(j\,\alpha)_i}{i!}\int_0^{y} x^{r+i}(1+x)\mathrm{e}^{-\lambda\,(j\,\alpha+1)x}\mathrm{d}x. \tag{11}$$

On expressing the integral in Equation (11) in terms of the incomplete gamma function

$\gamma(a,y)=\int_0^y z^{a-1}\,\mathrm{e}^{-z}\mathrm{d}z$, we have

$$m_r(y)=\frac{\lambda^2}{1+\lambda}\sum_{i,j=0}^{\infty} K_{i,j}\left\{\frac{\gamma\,(r+i+1,(j\,\alpha+1)\lambda y)}{[(j\,\alpha+1)\lambda]^{r+i+1}}+\frac{\gamma\,(r+i+2,(j\,\alpha+1)\lambda y)}{[(j\,\alpha+1)\lambda]^{r+i+2}}\right\}, \quad (12)$$

where $K_{i,j}=w_j\,[\lambda/(1+\lambda)]^i\,(j\,\alpha)_i/i!$ for $i,j=0,1\ldots$.

Bonferroni and Lorenz curves as well as mean deviations can be determined by letting $r=1$ in Equation (12). The Bonferroni and Lorenz curves are defined (for a given probability π) as $B(\pi)=m_1(q)/(\pi\,\mu_1')$ and $L(\pi)=m_1(q)/\mu_1'$, respectively, where $q=Q(\pi)$ may be established from Equation (6). The mean deviations about the mean and about the median are given by $\delta_1=\mathrm{E}(|X-\mu_1'|)=2\mu_1'\,F(\mu_1')-2m_1(\mu_1')$ and $\delta_2=\mathrm{E}(|X-M|)=\mu_1'-2m_1(M)$, where the median M and the mean μ_1' can be evaluated from Equations (6) and (10), respectively. We now provide a general formula for $M(t)=\mathrm{E}(\mathrm{e}^{t\,X})$, the MGF of X. The MGF of the gamma PDF with parameters α and β is $(1-t/\beta)^{-\alpha}$ $(t<\beta)$. Then, if follows from Equation (9) that, for $t<\lambda$,

$$M(t)=\sum_{i,j=0}^{\infty}\left[v_{i,j}^{(1)}\left(1-\frac{t}{(j\,\alpha+1)\lambda}\right)^{-i-1}+v_{i,j}^{(2)}\left(1-\frac{t}{(j\,\alpha+1)\lambda}\right)^{-i-2}\right].$$

The last aspect being discussed in this section is the distribution of order statistics. Order statistics appear in many areas of statistical theory and practice. Suppose $X_1,\ldots,X_n$ is a random sample from the HEL distribution and let $X_{i:n}$ denote the ith order statistic. The PDF of $X_{i:n}$ can be expressed as

$$f_{i:n}(x)=K\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}f(x)\,F(x)^{k+i-1}, \quad (13)$$

where $K=1/B(i,n-i+1)$ and $B(p,q)=\Gamma(p)\Gamma(q)/\Gamma(p+q)$ is the beta function.

Consider the following representation available from Gradshteyn and Ryzhik (2000) for a power series raised to a positive integer n:

$$\left(\sum_{j=0}^{\infty}a_j\,u^j\right)^n=\sum_{j=0}^{\infty}b_{n,j}\,u^j, \quad (14)$$

where the coefficients $b_{n,j}$, for $n=1,2,\ldots$ and $j=1,2,\ldots$, are obtained from the recursive equation

$$b_{n,j}=(j\,a_0)^{-1}\sum_{m=1}^{j}[m(n+1)-j]\,a_m\,b_{n,j-m},$$

with $b_{n,0}=a_0^n$. On integrating the right-hand side of Equation (7), we can write

$$F(x)=\bar{G}_{\mathrm{L}}(x)\sum_{j=0}^{\infty}w_j\,\bar{G}_{\mathrm{L}}(x)^{j\,\alpha},$$

and then making use of Equation (14), we have

$$F(x)^{k+i-1} = \sum_{j=0}^{\infty} t_{k+i-1,j}\, \bar{G}_{\rm L}(x)^{j\,\alpha+k+i-1},$$

where $t_{k+i-1,j} = (j\,w_0)^{-1} \sum_{m=1}^{j} [m(k+i) - j]\, w_m\, t_{k+i-1,i-m}$ for $j \geq 1$ and $t_{k+i-1,0} = w_0^{k+i-1}$. Inserting the previous expression for $F(x)^{k+i-1}$ and the representation of Equation (7) of the PDF appearing in Equation (13) gives

$$f_{i:n}(x) = K \sum_{r,j=0}^{\infty} \sum_{k=0}^{n-i} v_{r,j,k} h_{(r+j)\,\alpha+k+i}(x), \tag{15}$$

where

$$v_{r,j,k} = \frac{(-1)^k\,(r\,\alpha+1)\; w_r\; t_{k+i-1,j}}{(r+j)\,\alpha+k+i} \binom{n-i}{k}.$$

Equation (15) reveals that the PDF of the HEL order statistics can be expressed as a triple linear combination of LIIL PDFs. Accordingly, certain mathematical properties of the HEL order statistics could be determined from those of the LIIL distribution.

4. Parameter Estimation

We now discuss the estimation of the model parameters using the ML method. There exist several approaches for estimating parameters; however, the ML method is the most commonly employed. The ML estimators enjoy several desirable properties and can be utilized in the construction of confidence intervals for the model parameters. They also appear in some test statistics. The normal approximation to the distribution of these estimators follows from large sample distribution theory.

Let $X_1, \ldots, X_n$ be a sample of size n from the HEL distribution whose associated PDF is given in Equation (5). The log-likelihood function $\ell = \ell(\Theta)$ of the vector of parameters $\Theta = (\theta, \alpha, \lambda)^\top$ is given by

$$\ell = \frac{n}{\alpha}\,\log\,\theta + n\,\log\left(\frac{\lambda^2}{1+\lambda}\right) + \sum_{i=1}^{n} \log(1+x_i) - \lambda\,x_i - (1+\frac{1}{\alpha}) \sum_{i=1}^{n} \log[1-\bar{\theta}\bar{G}_{\rm L}(x)^{\alpha}]. \tag{16}$$

The ML estimates $\widehat{\theta}$, $\widehat{\alpha}$ and $\widehat{\lambda}$ are determined by maximizing the log-likelihood function of Equation (16) with respect to the parameters θ, α and λ. In general, there is no closed-form representation for these estimates, which are determined in practice the by making use of numerical methods. Equation (16) can be maximized either directly by using the R (`optim function`), SAS (`NLMixed` procedure) or Ox (`MaxBFGS` function), or by solving the nonlinear likelihood equations obtained by equating the partial derivatives of ℓ with respect to each parameter to zero.

The components of the score vector $U(\Theta)$ are expressed as

$$U_\theta = \frac{n}{\alpha\,\theta} - \left(1+\frac{1}{\alpha}\right)\sum_{i=1}^{n}\frac{\bar{G}_{\rm L}(x)^\alpha}{1-\bar{\theta}\bar{G}_{\rm L}(x)^\alpha},$$

$$U_\alpha = -\frac{n}{\alpha^2}\,\log\theta - \frac{1}{\alpha^2}\sum_{i=1}^{n}\frac{\bar{G}_{\rm L}(x)^\alpha\log\bar{G}_{\rm L}(x)}{1-\bar{\theta}\bar{G}_{\rm L}(x)^\alpha},$$

$$U_\lambda = \frac{n(2+\lambda)}{\lambda+\lambda^2} + x_i + \left(1+\frac{1}{\alpha}\right)\sum_{i=1}^{n}\left[\frac{\alpha\,\bar{\theta}\bar{G}_{\rm L}(x)^{\alpha-1}}{(1+\lambda)^2[1-\bar{\theta}\bar{G}_{\rm L}(x)^\alpha]}\,\lambda\,x_i\,[2+\lambda+(1+\lambda)x_i]\,{\rm e}^{-\lambda\,x_i}\right].$$

Setting these equations to zero and solving them simultaneously yields the ML estimates of the model parameters.

We now assess the performance of the ML estimators of the model parameters by means of Monte Carlo simulations. The simulations are replicated 1,000 times with samples of sizes $n = 50, 100, 200$ and the following parameter values: I: $\theta = 0.5$, $\alpha = 0.5$ and $\lambda = 1$; II: $\theta = 0.1$, $\alpha = 1.5$ and $\lambda = 1$; III: $\theta = 1.5$, $\alpha = 0.5$ and $\lambda = 1$; IV: $\theta = 1.5$, $\alpha = 1.5$ and $\lambda = 1$. Table 2 lists the average bias (Bias) of the ML estimators, mean squared errors (MSE), coverage probabilities (CP) and average widths (AW) of the confidence intervals for the parameters θ, α and λ and the three sample sizes. From these results, we conclude that the ML estimators perform well when it comes to estimating the parameters of the HEL distribution. In general, the biases, MSEs and AWs decrease when the sample size increases. Moreover, the CPs of the confidence intervals are quite close to the 95% nominal level. Thus, the ML estimators and their asymptotic distributional properties can be adopted for constructing approximate confidence intervals for the parameters of the HEL distribution.

5. Empirical Illustrations with Hydrological Data

In this section, we fit the HEL model and some other competing models to two hydrological data sets. We assess how well the HEL distribution performs as compared to the beta-Lindley (BL) studied by Mervoci and Sharma (2014), exponentiated power Lindley (EPL) due to Ashour and Eltehiwy (2015), beta-exponential (BE) proposed by Nadarajah and Kotz (2006), exponentiated Nadarajah and Haghighi (ENH) defined by Lemonte (2013), Harris extended exponential (HEE) discussed by Pinho et al. (2015), exponentiated Weibull (EW) studied by Mudholkar and Sharivastava (1993), power Lindley (PL) introduced by Ghitany et al. (2013), exponentiated Lindley defined by Nadarajah et al. (2007) and Lindley distributions. For each model, we estimated the parameters by the ML method and assessed the goodness-of-fit by means of the Akaike information criterion (AIC), Cramér-von Mises (W), Anderson-Darling (AD), Kolmogrov-Smirnov (KS) and average scaled absolute error (ASAE) statistics. The ASAE is defined as (see Castilo and Hadi, 2005)) ASAE $= (1/n)\sum_{i=1}^{n}(|x_{(i)} - \widehat{x}_{(i)}|)/(x_{(n)} - x_{(1)})$, where $x_{(i)}$ is the observed value of ith order statistic, and $\widehat{x}_{(i)}$ is obtained from the QF, $Q(u_i)$, wherein the ML estimates are substituted to the parameters, with $u_i = i/(n+1)$. The ASAE statistic is useful for measuring the accuracy of the fitted model. In general, the smaller values of the above statistics indicate a better fit to the data.

Table 2. Monte Carlo simulation results for the listed statistical indicator.

Parameter	n	Bias	MSE	CP	AW
		I			
θ	50	−0.044	0.112	0.92	1.483
	100	−0.037	0.045	0.95	0.979
	200	−0.037	0.033	0.98	0.749
α	50	0.626	1.690	0.96	2.079
	100	0.419	0.429	0.95	0.983
	200	0.314	0.110	0.95	0.799
λ	50	−0.028	0.193	0.93	1.459
	100	−0.042	0.111	0.96	1.154
	200	−0.046	0.079	0.95	0.123
		II			
θ	50	0.022	0.007	0.93	0.368
	100	0.012	0.003	0.96	0.232
	200	0.004	0.001	0.95	0.153
α	50	0.621	1.340	0.95	4.809
	100	0.199	0.537	0.95	2.475
	200	0.078	0.167	0.95	1.588
λ	50	0.162	0.293	0.91	2.117
	100	0.080	0.133	0.94	1.436
	200	0.026	0.063	0.95	0.994
		III			
θ	50	1.317	0.589	0.98	1.508
	100	0.609	0.371	0.98	1.192
	200	0.288	0.148	0.96	0.506
α	50	1.375	0.473	0.90	1.624
	100	0.563	0.171	0.98	1.270
	200	0.157	0.049	0.95	0.014
λ	50	0.264	0.479	0.91	1.006
	100	0.204	0.278	0.95	0.214
	200	0.199	0.130	0.96	0.102
		IV			
θ	50	0.638	3.602	0.90	2.835
	100	0.237	1.276	0.91	1.401
	200	0.141	0.629	0.94	0.038
α	50	−0.003	0.083	0.96	1.156
	100	0.015	0.042	0.96	0.818
	200	−0.001	0.021	0.95	0.571
λ	50	0.117	0.255	0.96	1.977
	100	0.035	0.104	0.96	1.323
	200	0.024	0.055	0.96	0.923

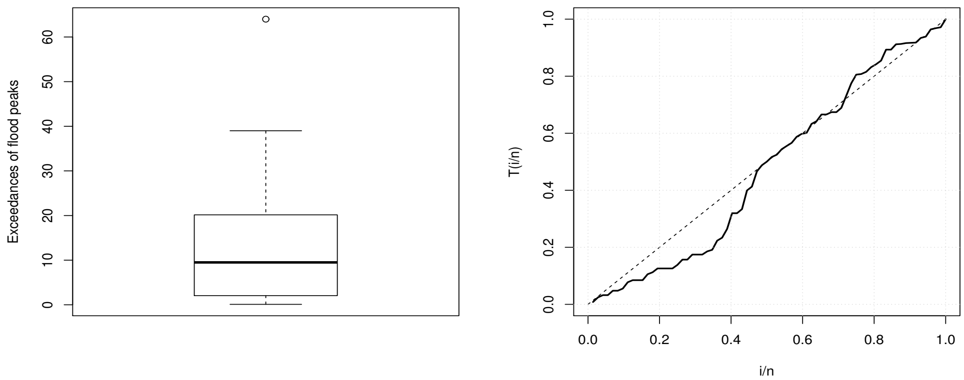
The CDFs of the BL, EPL, BE, ENH, HEE, EW, MOL, PL and EL distributions are given by

$$
\begin{aligned}
F_{\rm BL}(x,a,b,\theta) &= I_{1-(1+\frac{\theta\,x}{1+\theta})\mathrm{e}^{-\theta x}}(a,b), \quad x,\theta>0,\\
F_{\rm EPL}(x,\alpha,\beta,\theta) &= \left(1-(1+\tfrac{\theta\,x^{\beta}}{1+\theta})\mathrm{e}^{-\theta\,x^{\beta}}\right)^{\alpha}, \quad x,\alpha,\beta,\theta>0,\\
F_{\rm BE}(x,a,b,\lambda) &= I_{1-\mathrm{e}^{-\lambda\,x}}(a,b), \quad x,a,b,\lambda>0.\\
F_{\rm ENH}(x,\beta,\alpha,\lambda) &= \left(1-\mathrm{e}^{1-(1+\lambda\,x)^{\alpha}}\right)^{\beta}, \quad x,\beta,\alpha,\lambda>0,\\
F_{\rm HEE}(x,\beta,k,\lambda) &= \frac{\beta^{1/k}\mathrm{e}^{-\lambda\,x}}{[1-(1-\beta)\mathrm{e}^{-\lambda\,k\,x}]^{1/k}}, \quad x,\beta,k,\lambda>0,\\
F_{\rm EW}(x;c,\alpha,\lambda) &= \left(1-\mathrm{e}^{-(x/\lambda)^{c}}\right)^{\alpha}, \quad x,c,\alpha,\lambda>0,\\
F_{\rm MOL}(x,\alpha,\lambda) &= \frac{1-(1+\lambda)^{-1}[1+\lambda+\lambda\,x]\mathrm{e}^{-\lambda\,x}}{1-(1-\alpha)(1+\lambda)^{-1}[1+\lambda+\lambda\,x]\mathrm{e}^{-\lambda\,x}}, \quad x,\alpha,\lambda>0,\\
F_{\rm PL}(x,\beta,\theta) &= 1-(1+\tfrac{\theta\,x^{\beta}}{1+\theta})\mathrm{e}^{-\theta\,x^{\beta}}, \quad x,\beta,\theta>0,\\
F_{\rm EL} &= \left[1-\left(\frac{1+\theta+\theta\,x}{1+\theta}\right)\mathrm{e}^{-\theta x}\right]^{\alpha}, \quad x,\theta>0,
\end{aligned}
$$

respectively, where $I_z(p,q)$ denotes the incomplete beta function.

First, we consider a data set consisting of s exceedances (rounded to one decimal place) of flood peaks (in m^3/s) of the Wheaton river, which is located in the Yukon Territory, Canada, for the years 1958-1984. The data set is the following: 1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0. Some summary statistics of these data are: $n = 72$, $\bar{x} = 12.20417$, $s = 12.29722$, coefficient of skewness = 1.47251 and coefficient of kurtosis = 2.88955. The boxplot of these observations displayed in Figure 3(a) indicates that the distribution is right-skewed. The TTT (total time on test) plot (see, e.g., Gill, 1986; Aarset, 1987) of these data is shown in Figure 3(b). It is first convex and then concave, which suggests a bathtub-shaped failure rate. Accordingly, the HEL distribution could, in principle, be appropriate for modeling these data. The ML estimates (with the corresponding standard errors -SEs- in parentheses) as well as the ASAE, AIC, KS, CM and AD statistics are given in Table 3. All five goodness-of-fit statistics indicate that the HEL model provides the best fit. For a visual comparison, the empirical SF (ESF) and estimated SF associated with the HEL model as well as a theoretical versus empirical probability (PP) plot, which compares the empirical CDF of the data with the fitted CDF, are respectively included in Figures 4(a) and 4(b). Clearly, the HEL model closely fits the data distribution.

In this second illustration, the data set, which is freely available on the Korea Meteorological Administration (KMA) website (http://www.kma.go.kr), represents the annual maximum daily rainfall amounts in millimeters in Seoul (Korea) during the period 1961-2002. Some summary statistics of these precipitation data are: $n = 128$, $\bar{x} = 144.5991$, $s = 66.17812$, coefficient of skewness = 0.94067 and coefficient of kurtosis = 0.80435. The boxplot of these observations that is displayed in Figure 5(a) indicates that the distribution is right-skewed. The TTT plot appearing in Figure 5(b) suggests an increasing failure



(a) (b)

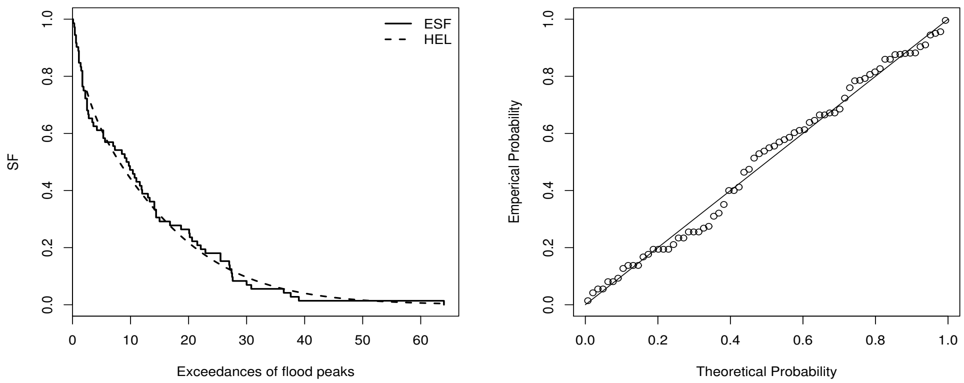
Figure 3. Boxplot (a) and TTT plot (b) for the flood data.

Table 3. ML estimates, SEs (in parentheses) and goodness-of-fit measures for the flood data.

Distribution	Estimates			ASAE	AIC	KS	CM	AD
HEL(θ, α, λ)	0.077	6.135	0.110	0.017	503.194	0.073	0.054	0.338
	(0.038)	(2.031)	(0.014)					
BL(a, b, θ)	0.556	0.275	0.334	0.020	510.206	0.115	0.126	0.775
	(0.098)	(0.241)	(0.273)					
EPL(α, β, θ)	0.916	0.730	0.300	0.025	510.425	0.106	0.149	0.857
	(0.595)	(0.235)	(0.279)					
BE(a, b, λ)	0.812	0.412	0.179	0.023	508.465	0.098	0.122	0.705
	(0.137)	(0.290)	(0.131)					
ENH(β, α, λ)	0.732	1.675	0.032	0.019	507.850	0.106	0.104	0.632
	(0.137)	(0.143)	(0.032)					
HEE(β, k, λ)	0.433	5.086	0.071	0.023	506.460	0.078	0.094	0.550
	(0.193)	(0.147)	(0.011)					
EW(c, α, λ)	1.387	0.519	0.016	0.403	508.050	0.107	0.105	0.642
	(0.587)	(0.308)	(0.036)					
MOL(α, λ)	0.216	0.090		0.044	522.571	0.175	0.582	4.148
	(0.128)	(0.023)						
PL(β, θ)	0.700	0.339		0.026	508.444	0.105	0.154	0.877
	(0.057)	(0.056)						
EL(α, θ)	0.509	0.104		0.021	509.349	0.117	0.135	0.833
	(0.077)	(0.015)						
L(θ)	0.153			0.044	530.424	0.241	0.819	7.424
	(0.013)							

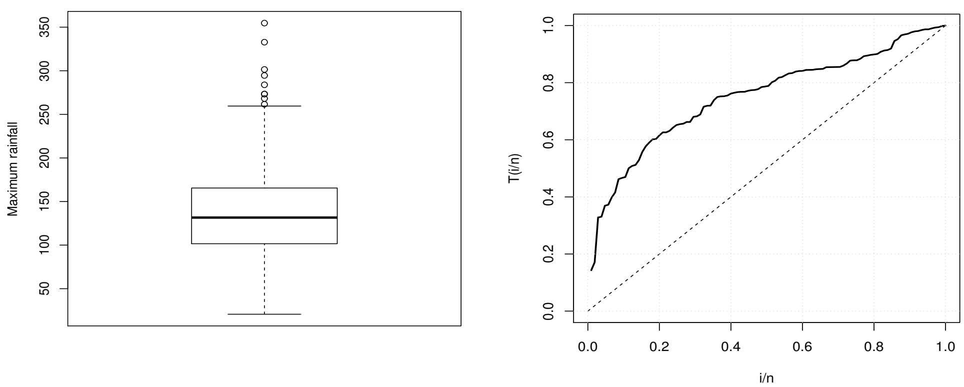
rate. The estimates of the parameters of the fitted distributions are listed in Table 4. We note that the HEL model has the lowest ASAE, AIC, KS, CM and AD values, which indicate that it provides the most accurate fit to the data. Furthermore, the ESF and estimated SF and PP plots shown in Figures 6(a) and 6(b) also suggest a close fit to the data distribution.

A likelihood ratio test can be utilized to compare a distribution having additional parameters with some of its sub-models. Accordingly, we made use of the likelihood ratio test to assess the improvement in fit that the HEL distribution produces with respect to



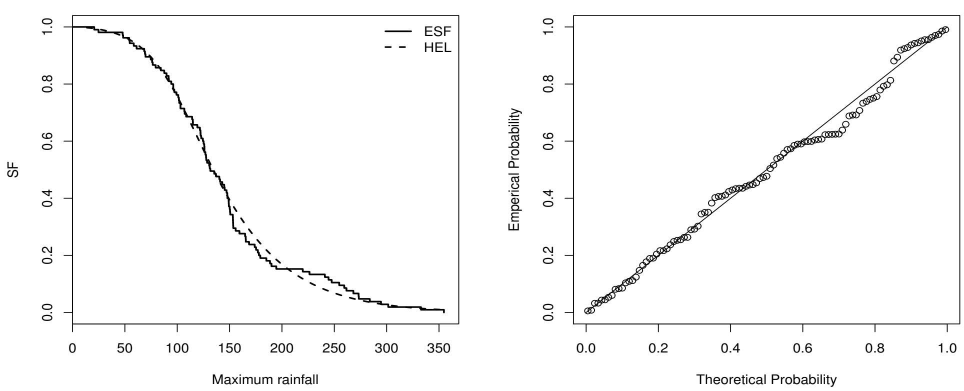
(a) (b)

Figure 4. Empirical SF and estimated HEL SF (a) and PP plot (b) for the flood data.



(a) (b)

Figure 5. Boxplot (a) and TTT plot (b) for the precipitation data.



(a) (b)

Figure 6. Empirical SF and estimated HEL SF (a) and PP plot (b) for the precipitation data.

Table 4. ML estimates, SEs (in parentheses) and goodness-of-fit measures for the precipitation data.

Distribution	Estimates			ASAE	AIC	KS	CM	AD
HEL(θ, α, λ)	17.443	3.081	0.022	0.019	1165.064	0.077	0.077	0.490
	(9.276)	(1.069)	(0.003)					
BL(a, b, θ)	2.776	1.117	0.020	0.022	1169.396	0.085	0.144	0.809
	(0.622)	(0.577)	(0.007)					
EPL(α, β, θ)	1.530	1.318	0.003	0.024	1168.717	0.097	0.150	0.862
	(0.225)	(0.025)	(0.004)					
BE(a, b, λ)	4.433	1.448	0.012	0.029	1172.022	0.092	0.263	1.412
	(0.685)	(0.535)	(0.003)					
ENH(β, α, λ)	4.183	1.694	0.006	0.024	1168.620	0.095	0.146	0.837
	(0.687)	(0.217)	(0.001)					
HEE(β, k, λ)	1.535	1.860	0.008	0.137	1241.535	0.276	2.569	13.078
	(0.299)	(0.847)	(0.001)					
EW(c, α, λ)	1.411	2.907	98.866	0.433	1168.586	0.093	0.142	0.821
	(0.334)	(1.519)	(29.851)					
MOL(α, λ)	10.455	0.029		0.032	1171.003	0.103	0.184	1.330
	(4.118)	(0.003)						
PL(β, θ)	0.014	16.182		1.433	4820.512	0.999	34.999	1631.130
	(0.007)	(2.037)						
EL(α, θ)	2.871	0.022		0.022	1167.600	0.084	0.146	0.818
	(0.501)	(0.002)						
L(θ)	0.014			0.584	1199.216	1.187	0.519	6.508
	(0.001)							

the Lindley and MOL distributions. It is known that, under the null hypothesis,

$$-2\log\left(\frac{\text{likelihood under the null hypothesis}}{\text{likelihood in the whole parameter space}}\right) \sim \chi^2(d),$$

where, asymptotically, $\chi^2(d)$ follows a chi-square distribution having d degrees of freedom, d being equal to the number of additional parameters in the extended model. Using this result and standard statistical tables, we can obtain critical values for the test statistic. Table 5 includes the likelihood ratio statistics and corresponding p-values for the two data sets. Given the values of these statistics and their associated p-values, we reject the null hypotheses for both data sets and conclude that the HEL model provides a significantly better representation of the distribution of these data than the Lindley or MOL distributions. The 95% bootstrap confidence intervals obtained for the parameters θ, α and λ are given in Table 6.

Table 5. Likelihood ratio statistics and their p-values.

Hypothesis	Flood data	Precipitation data
H_0: α=1 (MOL) H_1: $\alpha \neq 1$ (HEL)	21.377 (< 0.000)	7.939 (0.005)
H_0: α=θ=1 (L) H_1: $\alpha \neq 1$, $\theta \neq 1$ (HEL)	31.229 (< 0.000)	38.151 (<0.000)

Table 6. 95% bootstrap confidence intervals for the parameters θ, α and λ.

Data set	θ	α	λ
Flood data	(0.039, 0.225)	(3.036, 10.429)	(0.087, 0.146)
Precipitation data	(8.243, 20.463)	(1.378, 5.027)	(0.018, 0.031)

Next, we present the concepts of return period, mean deviation about a return level and the rth moment of the order statistics. For a given a data set, the return period can be estimated by $\widehat{T} = 1/\bar{F}(x)$, where $\bar{F}(x) = 1 - F(x)$ and $F(x)$ denote the CDF of the distribution. The estimated return periods ($\widehat{T}$) correspond to the return levels (x_T) for each of these two data sets. They are reported in Table 7 and have been computed as $T = 1/\bar{F}(x_T)$, where $\bar{F}(\cdot)$ is as given in Equation (4). The mean deviation about a return level which is the mean of the distances of the values from their return level is given by $\eta = 2\, x_T F(x_T) - x_T - \mu + 2 \int_{x_T}^{\infty} x\, f(x)\, \mathrm{d}x$, where $f(\cdot)$ and $F(\cdot)$ denote the HEL PDF and CDF. Table 7 provides the mean deviations about certain values of the return levels ($\bar{x}_T$) for both the flood and precipitation data sets.

Table 7. Estimated return periods ($\widehat{T}$) and mean deviations about the return levels (η).

Flood data			Precipitation data		
x_T	$\widehat{T}$	η	x_T	$\widehat{T}$	η
140	499147.836	127.800	410	315.215	265.623
100	8350.571	87.802	375.5	160.422	435.000
50	62.48360	38.135	315.5	50.389	172.849
30	10.375	19.949	260	17.693	121.247
10	2.265	9.337	210	7.093	80.513

In order to be able to plan for future emergencies in connection with various hydrological events, it is useful to ascertain some distributional results on certain of the order statistics. To that end, we determine the rth moment, for $r = 1, 2, 3, 4$, of some order statistics for each data sets under the HEL model wherein the parameters are replaced by their ML estimates. Those moments are included in Table 8 for each data set.

Table 8. Some numerical values of $\mathrm{E}(X_{i:n}^r)$ for the indicated data set.

Flood data			Precipitation data		
i	r	$\mathrm{E}(X_{i:72}^r)$	i	r	$\mathrm{E}(X_{i:128}^r)$
1	1	0.097	1	1	21.409
	2	0.019		2	585.869
	3	0.006		3	18628.800
	4	0.002		4	658641.210
20	1	2.868	15	1	77.111
	2	8.962		2	5989.380
	3	30.433		3	468486.450
	4	111.999		4	3.689×10^4
60	1	22.898	30	1	98.427
	2	543.677		2	9719.320
	3	12726.600		3	962824.794
	4	308653.083		4	9.568×10^7

6. Concluding Remarks

We introduced a three-parameter extension of the Lindley distribution refereed to as the Harris extended Lindley (HEL) distribution, which is obtained by applying the Harris extended method to the Lindley distribution. The proposed model has two shape parameters and one scale parameter. It includes as sub-models the Marshall-Olkin Lindley and Lindley distributions. The HEL PDF can be decreasing or unimodal. Moreover, the HEL HR can be increasing, decreasing, unimodal (upside-down bathtub) or bathtub-shaped. We gave explicit expressions for the ordinary and incomplete moments, mean deviations, Bonferroni and Lorenz curves and order statistics associated with the proposed distribution. The estimation of the model parameters was successfully carried out by making use of the maximum likelihood method. In conclusion, the HEL distribution provides a very flexible model for fitting the wide spectrum of positive data sets arising in engineering, survival analysis, hydrology, economics, biology as well as numerous other fields of scientific investigation. All the calculations were performed with the symbolic computing software Mathematica, the code being available from the authors upon request.

Acknowledgements

We would like to express our sincere gratitude to the referees, an Associate Editor and the Editor-in-Chief for their valuable comments and suggestions which significantly improved the presentation of the results. The financial support of the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged by the the third author.

References

Aarset, M.V., 1987. How to identify bathtub hazard rate. IEEE Transactions on Reliability, 36, 106-108.

Alizadeh, M., MirMostafaee, S.M.T.K., and Ghosh., I., 2017. A new extension of power Lindley distribution for analyzing bimodal data. Chilean Journal of Statistics, 8, 67-86.

Aly, E.A.A. and Benkherouf, L., 2011. A new family of distributions based on probability generating functions. Sankhyā B, 73, 70-80.

Ashkar, F. and El Adlouni, S., 2014. Correcting confidence intervals for quantiles of a heavy-tailed distribution: Case study of two-parameter kappa distribution. Journal of Hydrology, 512, 498-505.

Ashour, S.K. and Eltehiwy, M.A., 2015. Exponentiated power Lindley distribution. Journal of Advance Research, 6, 895-905

Bhunya, P.K., Singh, R.D., Berndtsson, R., and Panda, S.N., 2012. Flood analysis using generalized logistic models in partial duration series, Journal of Hydrology, 420, 59-71.

Castillo, E., Hadi, A.S., Balakrishnan, N., and Sarabia, J.M., 2005. Extreme Value and Related Models with Applications in Engineering and Science, Wiley, Hoboken.

Chadwick, A., Morfett, J., and Borthwick, M., 2004. Hydraulics in Civil and Environmental Engineering, Spon Press, London.

Cordeiro, G.M., Alizadeh, M., Nascimento, A.D., and Rasekhi, M., 2016. The exponentiated Gompertz generated family of distributions: Properties and applications. Chilean Journal of Statistics, 7, 29-50

Corless, R.M., Gonnet, G.H., Hare, D.E.G., Jeffrey, D.J., and Knuth, D.E., 1996. On the Lambert W function. Advances in Computational Mathematics, 5, 329-359.

Ghitany, M.E., Al-Mutairi, D.K., and Aboukhamseen, S.M., 2015. Estimation of the reliability of a stress-strength system from power Lindley distributions. Communications in Statistics - Theory and Methods, 44, 118-136.

Ghitany M.E., Al-Mutairi D.K., Al-Awadhi, F.A., and Al-Burais M.M., 2012. Marshall-Olkin extended Lindley distribution and its application. International Journal of Applied Mathematics, 25, 709-721.

Ghitany, M.E., Al-Mutairi, D.K., Balakrishnan, N., and Al-Enezi, L.J., 2013. Power Lindley distribution and associated inference. Computational Statistics and Data Analysis, 64, 20-33.

Ghitany, M.E., Alqallaf, F., Al-Mutairi, D.K. and Husain, H.A., 2011. A two-parameter weighted Lindley distribution and its applications to survival data. Mathematics and Computers in Simulation, 81, 1190-1201.

Ghitany, M.E., Atieh, B., and Nadarajah, S., 2008. Lindley distribution and its application. Mathematics in Computer and Simulation, 78, 493-506.

Gill, D.R. (1986). The total time on test plot and the cumulative total time on test statistic for a counting process. The Annals of Statistics, 14, 1234-1239.

Gradshteyn, I.S. and Ryzhik, I.M., 2000. Tables of Integrals, Series, and Products, Academic Press, New York.

Harris, T.E., 1948. Branching processes. Annals of Mathematical Statistics, 19, 474-494.

Heo, J. and Boes, D.C., 2011. Regional flood frequency analysis based on a Weibull model: Part 2. Simulations and applications. Journal of Hydrology, 242, 171-182.

Jodrá, J., 2010. Computer generation of random variables with Lindley or Poisson-Lindley distribution via the Lambert W function. Mathematics and Computers in Simulation, 81, 851-859.

Kang, D., Ko, K., and Huh, J., 2015. Determination of extreme wind values using the Gumbel distribution. Energy, 86, 51-58.

Lemonte, A.J., 2013. A new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. Computational Statistics and Data Analysis, 62, 149-170.

Li, C., Singh, V.P., and Mishra, A.K. 2013. A bivariate mixed distribution with heavy-tailed component and its application to single-site daily rainfall simulation. Water Resource Research. 49, 767-789.

Lindley, D.V., 1958. Fiducial distributions and Bayes theorem. Journal of Royal Statistical Society B, 20, 102-107.

Marshall, A.N. and Olkin, I., 1997. A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. Biometrika, 84, 641-652.

Mazucheli, J., Coelho-Barros, E.A., and Louzada, F., 2016. On the hypothesis testing for the weighted Lindley distribution. Chilean Journal of Statistics, 7, 17-27.

Mervoci, F. and Sharma, V.K., 2014. The beta Lindley distribution: Properties and Applications. Journal of Applied Mathematics, Art.ID 198951, 10 p.

Mudholkar, G.S. and Srivastava, D.K., 1993. Exponentiated Weibull family for analyzing bathtub failure data. IEEE Transactions on Reliability, 42, 299-302.

Nadarajah, S., Bakouch, H., and Tahmasbi, R., 2007. A generalized Lindley distribution. Sankhya B, 73, 331-359.

Nadarajah, S. and Kotz, S., 2006. The beta exponential distribution. Reliability Engineering and System Safety, 91, 689-697.

Pinho, L.G.B., Cordeiro, G.M., and Nobre, J.S., 2015. The Harris extended exponential distribution. Communications in Statistics - Theory and Methods, 44, 3486-3502.

Zelenhasic, E.F., 1970. Theoretical probability distributions for flood peaks. Hydrology papers (Colorado State University), ID 42.

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